Analytical solutions for pressure perturbation and fluid leakage through aquitards and wells in multilayered-aquifer systems

Abdullah Cihan,[1] Quanlin Zhou,[1] and Jens T. Birkholzer[1]

Received 29 March 2011; revised 22 August 2011; accepted 25 August 2011; published 7 October 2011.

[1] Large-scale groundwater pumping or deep fluid injection in a multilayered subsurface system may generate pressure perturbation not only in the target formation(s), but also in over- and underlying units. Hydraulic communication in the vertical direction may occur via diffuse leakage through aquitards and/or via focused leakage through leaky wells. Existing analytical solutions for pressure perturbation and fluid flow in such systems consider either diffuse leakage or focused leakage, but never in combination with each other. In this study, we developed generalized analytical solutions that account for the combined effect of diffuse and focused leakage. The new solutions solve for pressure changes in a system of N aquifers with alternating leaky aquitards in response to fluid injection/extraction with any number, N_I, of injection/pumping (active) wells, and passive leakage/recharge in any number, N_L, of leaky wells. The equations of horizontal groundwater flow in the aquifers are coupled by the vertical flow equations in the aquitards and by the flow continuity equations in the leaky wells. The solution methodology, described in detail in this paper, involves transforming the transient flow equations into the Laplace domain; decoupling the resulting ordinary differential equations (ODEs) coupled by diffuse leakage via eigenvalue analysis; solving a system of $N_L \times N$ linear algebraic equations for the unknown rates of flow through leakage wells; and superposing the solution of pressure buildup/drawdown in aquifers and aquitards resulting from flow in the N_I active and N_L leaky wells. Verification of the new methodology was achieved by comparison with existing analytical solutions for diffuse leakage and for focused leakage, and against a numerical solution for combined diffuse and focused leakage. Application to an eight-aquifer system with leaky aquitards and one leaky well demonstrates the usefulness and efficiency of the approach, and illustrates the pressure behavior over a spectrum of leakage scenarios and parameters.

Citation: Cihan, A., Q. Zhou, and J. T. Birkholzer (2011), Analytical solutions for pressure perturbation and fluid leakage through aquitards and wells in multilayered-aquifer systems, *Water Resour. Res., 47*, W10504, doi:10.1029/2011WR010721.

1. Introduction

[2] Sedimentary basins often consist of a number of relatively high-permeability aquifers alternating with relatively low-permeability aquitards. The aquifers and aquitards may often extend over large areas, even covering entire basins. Examples of large multilayered basins in the United States include the Dakota aquifer system in South Dakota [*Bredehoeft et al.*, 1983], the Cambrian-Ordovician aquifer system in the Illinois basin [*Young*, 1992; *Zhou et al.*, 2010], the coastal plain aquifer systems in Virginia [*Konikow and Neuzil*, 2007], and the Texas Gulf Coast basin [*Nicot*, 2008]. The alternating aquitards can play an important role in regional groundwater flow, because they frequently have non-negligible permeability and storativity. It has been shown, for example, that the comparably high aquitard storativity in the Dakota aquifer system significantly contributes to water production in the area [*Bredehoeft et al.*, 1983; *Konikow and Neuzil*, 2007]. Recently, *Konikow and Neuzil* [2007] assessed significant sources of groundwater stored in aquitards, which can be depleted and used for water supply. The nonzero permeability of aquitards has been demonstrated through laboratory core measurements [*Neuzil*, 1986, 1994; *Yang and Aplin*, 2007, 2010] and regional groundwater flow modeling [*Hart et al.*, 2006]; permeability values can vary over a large range, from 10^{-15} to 10^{-23} m^2 [e.g., *Neuzil*, 1994; *Yang and Aplin*, 2010].

[3] Many sedimentary basins with multiple aquifers and alternating aquitards have been affected by extensive drilling, e.g., for groundwater supply (mostly shallow units) [e.g., *Young*, 1992], as well as for oil/gas exploration and production (both shallow and deep units) [e.g., *Nordbotten et al.*, 2004; *Nicot*, 2009]. Production wells for groundwater supply may be simultaneously screened in multiple aquifers. Flow through such multiaquifer wells is believed to increase the effective regional-scale permeability of aquitards, which can be orders of magnitude higher than the permeability measured on individual cores [*Hart et al.*,

[1] Earth Sciences Division, Lawrence Berkeley National Laboratory, University of California, Berkeley, California, USA.

2006]. In oil/gas exploration and production, sealing or plugging of abandoned wells (e.g., unsuccessful exploration wells, once-producing wells in depleted reservoirs) is standard practice; however, older (historic) wells may be improperly sealed or cement plugs may have degraded over time. Therefore, abandoned wells are considered potential conduits for fluid leakage and groundwater contamination [*Gass et al.*, 1977, *Javandel et al.*, 1988; *Lesage et al.*, 1991], and they may provide another pathway for vertical communication in a multilayered system.

[4] Both diffuse leakage (through aquitards) and focused leakage (through leaky wells) can be particularly relevant in the case of strong and wide-spread pressure perturbations in the subsurface. Such perturbations may be caused by regional groundwater production, by oil and gas production, and by high-volume injection of fluids for liquid waste disposal or geologic sequestration of CO_2. The amount of CO_2, for example, that will need to be captured and stored in order for geologic sequestration to have a significant role in climate change mitigation is enormous [*Zhou and Birkholzer*, 2011]. A large coal-fired power plant may generate 5–10 million tons of CO_2 per year, which if injected underground can induce pressure buildup and brine displacement over large areas [e.g., *Nicot*, 2008; *Zhou et al.*, 2008, 2010; *Birkholzer and Zhou*, 2009]. *Birkholzer et al.* [2009] have demonstrated the importance of diffuse leakage in lowering pressure buildup ("pressure bleed-off") in the storage aquifer for aquitard permeabilities as low as 10^{-18} m^2. On the other hand, the possibility of focused leakage of CO_2 and brine through abandoned wells is one of the main environmental concerns for geologic sequestration of CO_2 in depleted oil reservoirs and saline aquifers [*Celia et al.*, 2004, 2011].

[5] As an alternative to numerical simulation models, analytical solutions are often employed in subsurface flow and transport applications because they are very efficient with regard to calculation time and do not require spatial discretization. Because the solutions can be obtained so fast, analytical methods can be very useful in sensitivity studies or uncertainty quantifications. They are particularly suitable when dealing with a large number of injection and leaky wells [e.g., *Celia et al.*, 2011], which, for numerical simulations, would require local mesh refinement around each well to assure accurate results. A variety of analytical solutions have been developed over the last 80 years or so for problems involving pumping/injection in multilayered-aquifer systems. The first category of such solutions allowed for diffuse leakage through aquitards, but did not consider leaky wells. In the 1930s through the 1960s, analytical solutions were developed for pumping from the lower aquifer of a two-aquifer–one-aquitard system. *Huisman and Kemperman* [1951] and *Hantush and Jacob* [1955] considered steady state (assuming zero aquitard storativity) flow, and *Hantush* [1960] considered transient flow (with aquitard storage) through the aquitard. The restriction of an infinite-volume top aquifer assumption in the transient solution was removed in the solution of *Neuman and Witherspoon* [1969], who assumed that the two-aquifer–one-aquitard system was bounded by impervious boundaries at the top and the bottom. These solutions were later improved, for example, to allow for pumping/injection wells of a large diameter with well-skin effects by *Moench* [1985], and to consider a laterally bounded aquifer–aquitard system by *Zhou et al.* [2009]. Analytical solutions were also extended to multilayered systems with any number of aquifers, e.g., for transient flow with zero aquitard storativity using both the Laplace transform and the Hankel transform [*Hemker*, 1985]; for steady state flow with zero aquitard storativity using the Fourier transform [*Maas*, 1987a]; and for transient flow with aquitard storativity using the Hankel and the Fourier transform [*Maas*, 1987b]. An important advancement in the solution methodology involved the use of eigenvalues and eigenvectors to decouple the system of ordinary differential equations (e.g., for steady state flow by *Hunt* [1985] and for transient flow in the Laplace domain by *Hemker and Maas* [1987]). This advancement allowed very efficient solutions for multilayered systems [*Cheng and Morohunfola*, 1993; *Hemker and Maas*, 1994; *Cheng*, 1994]. The solution method referred to above for diffuse leakage resulting from pumping/injection wells in multilayered systems was implemented into the commercial code MLU (Multi-Layer Unsteady State) [*Hemker and Post*, 2011]. *Veling and Maas* [2009] further improved the solution method for diffuse leakage in multilayered aquifers by including both horizontal and vertical flow components in aquifers and aquitards with partially penetrating wells.

[6] The second category of analytical solutions allowed for flow through leaky abandoned wells, but without consideration of diffuse leakage through aquitards. The development of these solutions was driven by the concern about leakage as potentially induced by fluid injection for liquid waste disposal and/or geologic carbon sequestration. *Javandel et al.* [1988] and *Avci* [1994] presented analytical solutions for leakage through one leaky well in a two-aquifer–one-aquitard system bounded by impervious boundaries at the top and the bottom. They presented their solutions as a convolution of (1) the unknown time-dependent leakage rate and (2) the existing fundamental solutions for either constant or instantaneous pumping/injection in a confined aquifer [*Theis*, 1935; *Bear*, 1979], and then applied the Laplace transform to solve for the leakage rate and the drawdown. *Javandel et al.* [1988] assumed that the top aquifer was of an infinite volume, while *Avci* [1994] allowed for pressure changes to occur in the bottom and top aquifers. Recently, *Nordbotten et al.* [2004] developed an analytical solution for leakage through multiple leaky wells in a multilayered system; their solution numerically evaluated the convolution integrals involving time-dependent leakage rates and approximated the exponential integral (i.e., the well function of infinite terms [*Theis*, 1935]) by its finite terms. This appeared to result in satisfactory accuracy except for very early-time behavior. Note that this solution needs adequate temporal discretization to obtain pressure changes and leakage rates at a given time, while the Javandel and Avci solutions can directly calculate these results, without any time-stepping. As evident from the above discussion, the existing analytical solutions either consider diffuse leakage through aquitards but no leaky wells, or they allow for focused leakage through leaky wells but not for diffuse leakage through aquitards. Simultaneous representation of both processes would allow for a more realistic prediction of the pressure and leakage behavior in natural multilayered aquifer–aquitard systems. Therefore, we describe here a set of new generalized analytical solutions for coupled diffuse and focused leakage in

a multilayered system consisting of any number of aquifers, alternating aquitards, pumping/injection wells, and leaky wells. These solutions encompass and advance the features and capabilities of the many existing analytical solutions discussed above and provide a major improvement in analytically solving the subsurface flow processes in multilayered aquifer–aquitard systems.

[7] This paper is organized as follows. Section 2 presents the groundwater flow equations for aquifers and aquitards and reviews the initial and boundary conditions, as well as the conditions at aquifer–aquitard interfaces, well–aquifer interfaces, and well–aquitard interfaces. In section 3, generalized analytical solutions are presented for systems with one injection well and one leaky well, starting with the solution procedure for diffuse leakage, followed by focused leakage, and finally combined leakage. Particular solutions for a two-aquifer–one-aquitard system are derived from the generalized solutions and compared with existing analytical solutions. The general solution for any number of injection wells and leaky wells is presented using superposition. The new solutions are verified in section 4 by comparison to existing analytical solutions and to numerical simulations. Finally, a "real-world" demonstration application with eight aquifers, seven aquitards, and one leaky well is presented in section 5. The new solutions were implemented into a FORTRAN-based software package described briefly in the appendix, which reads in the problem specifications, executes the calculation, and returns calculation results.

2. Governing Equations

[8] We consider a confined subsurface system consisting of any number of aquifers, alternating aquitards, pumping/injection wells, and leaky wells. The solution domain is bounded at the top and bottom by either no-flow or fixed-head boundary conditions, and extends infinitely in the horizontal direction. Initially, the subsurface system is at hydrostatic conditions, which are then perturbed by pumping and/or injection through multiple wells in selected aquifers at given rates. We are particularly interested in pressure perturbations that are strong enough to affect a large portion of the vertical sequence of aquifers and aquitards, which may be coupled with each other due to diffuse leakage through aquitards and focused leakage through leaky wells. We use injection activities as the example case for the following mathematical derivations, but point out that the solution methodology is also applicable to system perturbations induced by pumping alone or by combined pumping and injection. The analytical solutions calculate the transient behavior of pressure buildup in all aquifers and aquitards, the rate of diffuse leakage through aquitards, and the rate of focused leakage through leaky wells.

2.1. Problem Description and Assumptions

[9] We have made the following assumptions and notations (see Figure 1). Each of the N aquifers (aquifer i, $i = 1, \ldots, N$, numbered from the bottom aquifer to the top aquifer) is homogeneous and isotropic with hydraulic conductivity K_i (L T^{-1}) and specific storativity $S_{s,i}$ (L^{-1}); it is also assumed that each aquifer i has uniform thickness B_i (L). Different aquifers may have different hydrogeological properties (see Figure 1a). The $N + 1$ alternating aquitards (aquitard i, $i = 0, \ldots, N$), globally numbered from 0 at the bottom to N at the top, are also homogeneous and isotropic (though each may have different hydraulic conductivity K_i' and specific storativity $S'_{s,i}$) with uniform thickness B_i'. (Note that the specific example shown in Figure 1 has the vertical domain enveloped by top and bottom aquitards;

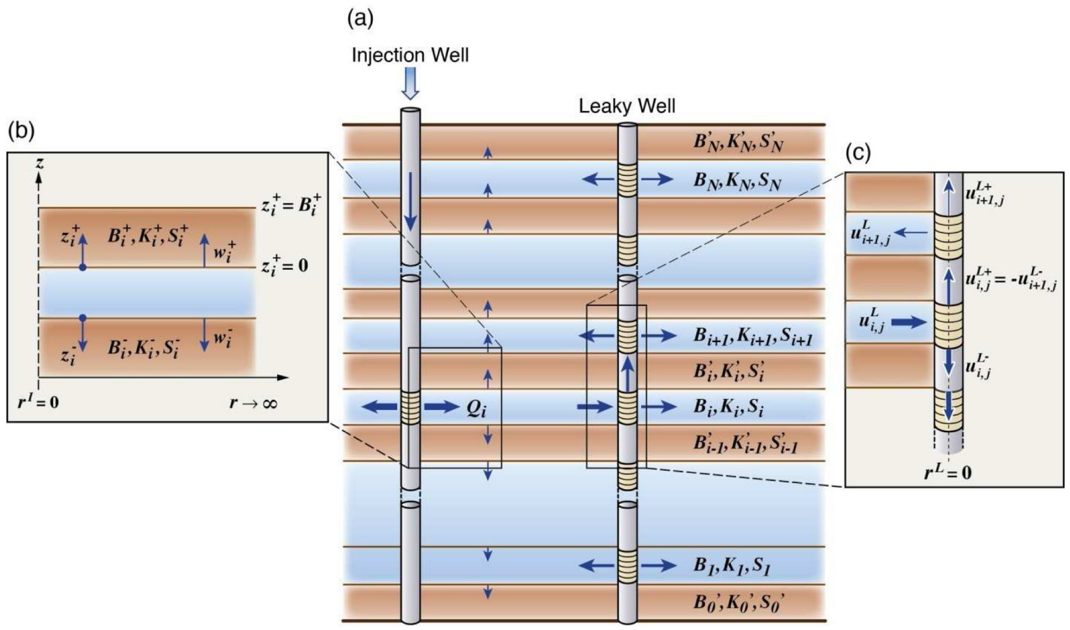

Figure 1. Description of (a) a typical multilayered system of N aquifers and $N + 1$ aquitards, with one injection well and one leaky well, (b) the local vertical coordinates for an aquifer and its neighboring aquitards, and (c) the continuity of well leakage. The length and thickness of arrows conceptually reflect the magnitude of diffuse- and focused-leakage rates.

however, any other configurations can be handled with the analytical solutions.) For solution development, we define a local aquitard numbering system relative to each aquifer i with the overlying ($\alpha = +$) or underlying ($\alpha = -$) aquitards denoted by a relative index (i, α), such that their properties are referred to as hydraulic conductivity K_i^α, specific storativity $S_{s,i}^\alpha$, and thickness B_i^α (see Figure 1b). It is assumed that groundwater flow is horizontal in the aquifers and vertical in the aquitards. This assumption is valid as long as the ratio of hydraulic conductivity between the aquifers and the aquitards is larger than 100, as demonstrated by previous studies [e.g., *Neuman and Witherspoon*, 1969].

[10] The new generalized solution allows for consideration of multiple injection wells, and each well may have multiple well intervals screened in selected aquifers. The rate, $Q_{i,m}$ ($L^3\ T^{-1}$), of fluid injection into aquifer i at injection well m ($m = 1, \ldots, N_I$, where N_I is the number of injection wells) is known, and thus can be assigned as a boundary condition. The radius of injection well m in aquifer i is $r_{wi,m}$. Multiple leaky wells may also exist in the aquifer–aquitard system. These leaky wells allow for fluid exchange between some or all of the aquifers of the multilayered system (Figure 1c). For example, the pressure buildup induced by fluid injection into aquifer i will drive native fluid into the wellbore (with a radius of $r_{Li,l}$) at leaky well l ($l = 1, \ldots, N_L$ where N_L is the number of leaky wells), and the fluid will then be transported via the wellbore into the overlying aquifer(s) and/or the underlying aquifer(s). The fluid exchange between aquifer i and well l is horizontal through the fully penetrated well–aquifer segment $L_{i,l}$, with an unknown leakage rate $u_{i,l}$ ($L^3\ T^{-1}$) (the solution also allows for $u_{i,l} = 0$ representing a cased, or impervious well–aquifer segment).

[11] Each leaky well segment may be assigned a different value for hydraulic conductivity (a well segment is defined as the vertical distance associated with the aquifers and aquitards that the leaky well penetrates). The rate of leakage from aquifer i through well l is the sum of the rates of leakage through the overlying and underlying well–aquitard segments, assuming there is no storage effect in the wellbore segment. The leakage rate through the overlying (or underlying) well–aquitard segment depends on the hydraulic head difference between aquifer i and aquifer $i + 1$ (or $i - 1$) and the hydraulic conductivity, $K_{Li,l}^\alpha$ ($L\ T^{-1}$), of well–aquitard segment $L_{i,l}^\alpha$, as well as the wellbore area ($\pi r_{Li,l}^2$) (L^2). The water that has leaked upward through well–aquitard segment $L_{i,l}^+$ may be diverted into the overlying aquifer $i + 1$ and/or may continue to migrate upward through the overlying well–aquitard segment, and beyond.

[12] In a thick multilayered system (which may extend from the ground surface to aquifers several kilometers deep), fluid properties of interest such as density or viscosity vary as a function of pressure, temperature, and salinity changes. For example, in situ groundwater density may vary by as much as 20% between near-surface conditions and conditions a few kilometers deep. We assume that the changes in density or viscosity distribution as a result of injection-induced pressurization or salinity changes from vertical leakage are negligible.

2.2. Governing Equations for Aquifers and Aquitards

[13] We start by presenting the governing equations for flow in aquifers and aquitards caused by injection through a *single* well (or leakage through a single leaky well). Superposition will be used in the following section to solve for pressure buildup and diffuse leakage rates in a system with *multiple* injection and leaky wells. The governing equation for single-phase radial flow in aquifer i is written in terms of hydraulic head buildup $s_i = s_i(r, t)$ (L) [*Bear*, 1972; *Moench*, 1985; *Hemker*, 1985; *Hunt*, 1985; *Maas*, 1987b; *Cheng and Morohunfola*, 1993; *Zhou et al.*, 2009]

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s_i}{\partial r}\right) = \frac{1}{D_i}\frac{\partial s_i}{\partial t} + \frac{w_i^-}{T_i} + \frac{w_i^+}{T_i}; \quad i = 1, \ldots, N, \tag{1a}$$

where $s_i = h_i(r, t) - h_{i0}$, h_i (L) is the hydraulic head in aquifer i, h_{i0} is the initial uniform head in the aquifer, D_i ($= K_i/S_{s,i}$) ($L^2\ T^{-1}$) is the hydraulic diffusivity, and T_i ($= K_iB_i$) ($L^2\ T^{-1}$) is the transmissivity, r (L) is the radial distance from the center of the well, and t (T) is the time. w_i^α ($L\ T^{-1}$) denotes the rate of diffuse leakage (i.e., specific discharge) through the aquifer–aquitard interface from aquifer i into the overlying ($\alpha = +$) or underlying ($\alpha = -$) aquitard, and can be calculated using

$$w_i^\alpha = -\frac{K_i^\alpha}{B_i^\alpha}\frac{\partial s_i^\alpha}{\partial z_{Di}^\alpha}\bigg|_{z_{Di}^\alpha=0}, \tag{1b}$$

where $s_i^\alpha = s_i^\alpha(r,\ z_{Di}^\alpha,\ t)$ (L) is the hydraulic head buildup in aquitard (i, α), z_{Di}^α $(= z_i^\alpha/B_i^\alpha,\ 0 \le z_{Di}^\alpha \le 1)$ is the dimensionless local vertical coordinate, and z_i^α (L) is the local vertical coordinate, with $z_i^\alpha = 0$ at the interface between aquifer i and aquitard (i, α) and $z_i^\alpha = B_i^\alpha$ at the interface between aquifer $i + \alpha$ and aquitard (i, α) (see Figure 1b). Note that $i + \alpha = i + 1$ for $\alpha = +$, while $i + \alpha = i - 1$ for $\alpha = -$.

[14] The one-dimensional vertical flow through aquitard (i, α) is written as,

$$\frac{\partial^2 s_i^\alpha}{\partial {z_{Di}^\alpha}^2} = \frac{\left(B_i^\alpha\right)^2}{D_i^\alpha}\frac{\partial s_i^\alpha}{\partial t}; \qquad 0 \le z_{Di}^\alpha \le 1, \tag{2a}$$

with the boundary conditions at aquifer–aquitard interfaces:

$$\begin{aligned} s_i^\alpha(r,\ 0,\ t) &= s_i(r, t), \\ s_i^\alpha(r,\ 1,\ t) &= s_{i+\alpha}(r, t), \end{aligned} \tag{2b}$$

where D_i^α $(= K_i^\alpha/S_{s,i}^\alpha)$ is the hydraulic diffusivity of aquitard (i, α), and there exists a relationship $s_i^+(r, z_{Di}^+, t) = s_{i+1}^-(r, z_{Di+1}^-, t)$ for $z_{Di+1}^- = 1 - z_{Di}^+$.

[15] The entire system of aquifers and aquitards is assumed to be at hydrostatic pressure initially. (This assumption is not necessary to develop the following analytical solutions for head buildup and leakage rates, but it helps simplify the solution development for the diffuse leakage case because initial head differences in the aquifers make equations [1] and [2] a coupled system of nonhomogeneous equations.) The initial conditions for the aquifers and the aquitards can be written as follows:

$$s_i(r,\ t = 0) = 0, \tag{3a}$$

$$s_i^\alpha\left(r, z_{Di}^\alpha,\ t = 0\right) = 0. \tag{3b}$$

[16] The outer lateral boundary is assumed to be far away from the perturbed region, i.e., at infinity, so that the boundary conditions become:

$$\begin{aligned} s_i(\infty, t) &= 0, \\ s_i^{\alpha}\left(\infty, z_{Di}^{\alpha}, t\right) &= 0. \end{aligned} \tag{4}$$

[17] The top and the bottom of the system may have either a zero head buildup or a no-flow condition:

$$s_1^-(r, 1, t) = 0 \text{ or} \tag{5a}$$

$$\partial s_1^-(r, 1, t)/\partial z_{D1}^- = 0, \tag{5b}$$

$$s_N^+(r, 1, t) = 0 \text{ or} \tag{6a}$$

$$\partial s_N^+(r, 1, t)/\partial z_{DN}^+ = 0. \tag{6b}$$

[18] Equations (1)–(6) are the general governing equations for groundwater flow in aquifers and aquitards in a confined multilayered system with one injection or one leaky well. The one-dimensional radial flow equations for aquifers are coupled with each other through the vertical flow in aquitards. In the presence of leaky wells, the groundwater flow in aquifers is also coupled through leakage via leaky wells. The boundary conditions at the injection and leaky wells are presented below.

2.3. Boundary Conditions for Active and Leaky Wells

[19] We define a well with a known injection rate (or a known pumping rate) as an active well, and a well with an unknown flow rate driven internally by hydraulic head gradients through well–aquifer segments as a leaky (passive) well [*Nordbotten et al.*, 2004]. For an active (injection) well, the injection rate may be constant or time-dependent, and the boundary condition at its radial wall of the cylindrical well interval screened in aquifer i is given by

$$-2\pi r_{wi,m} T_i \frac{\partial s_{i,m}^I\left(r_{wi,m}, t\right)}{\partial r} = Q_{i,m}(t), \tag{7}$$

where $Q_{i,m}(t)$ is the injection rate through injection well m into aquifer i, and $s_{i,m}^I$ is the head buildup in aquifer i caused by injection at well m. Note that $Q_{i,m}(t) > 0$ for injection.

[20] For a leaky well (leaky well l), the flow rate between aquifer i and the well is unknown and depends on the hydraulic head within the well segment, which is affected by the conditions in the two well–aquitard segments ($L_{i,l}^+$ and $L_{i,l}^-$) overlying and underlying the well–aquifer segment ($L_{i,l}$). In turn, the head buildup in all aquifers (as well as in the aquitards) depends on the unknown well leakage rate. For leaky well l, the boundary condition at well–aquifer segment $L_{i,l}$ is given by

$$-2\pi r_{Li,l} T_i \frac{\partial s_{i,l}^L\left(r_{Li,l}, t\right)}{\partial r} = u_{i,l}(t), \tag{8a}$$

where $s_{i,l}^L$ is the hydraulic head buildup in response to the unknown well leakage rate $u_{i,l}(t)$. Note that $u_{i,l}(t) < 0$ for leakage from the aquifer and $u_{i,l}(t) > 0$ for recharge into the aquifer. The vertical flow through a well–aquitard segment, $L_{i,l}^{\alpha}$, can be written as,

$$u_{i,l}^{\alpha} = -\frac{1}{\Omega_{i,l}^{\alpha}}\left(s_{i,l}^T - s_{i+\alpha,l}^T\right), \quad \Omega_{i,l}^{\alpha} = \frac{B_i^{\alpha}}{K_{Li,l}^{\alpha}\,\pi\left(r_{Li,l}^{\alpha}\right)^2}, \tag{8b}$$

where $s_{i,l}^T$ is the total head buildup (transient actual head buildup) evaluated at the leaky well under the effects of all active and leaky wells, and $\Omega_{i,l}^{\alpha}$ (T L^{-2}) is the resistance to flow through the well–aquitard segment. The continuity equation for the net leakage rate into or out of aquifer i through a leaky well connecting three consecutive aquifers ($i-1$, i, $i+1$) (see Figure 1c) is given by

$$\begin{aligned} u_{i,l} &= u_{i,l}^+ + u_{i,l}^- = \frac{1}{\Omega_{i,l}^-}\left(s_{i-1,l}^T - s_{i,l}^T\right) - \frac{1}{\Omega_{i,l}^+}\left(s_{i,l}^T - s_{i+1,l}^T\right), \\ u_{i,l}^+ &= -u_{i+1,l}^-. \end{aligned} \tag{8c}$$

[21] For a cased wellbore having fluid exchange only with aquifer i at the bottom and aquifer $i+k$ at the top, equation (8b) can be rewritten as

$$\begin{aligned} u_{i,l}^{+k} &= -\frac{1}{\Omega_{i,l}^{+k}}\left(s_{i,l}^T - s_{i+k,l}^T\right), \\ \Omega_{i,l}^{+k} &= \sum_{j=i}^{i+k-1} B_j^+/(K_{Lj,l}^+ \pi {r_{Lj,l}^+}^2) + \sum_{j=i+1}^{i+k-1} B_j/(K_{Lj,l}\pi r_{Lj,l}^2), \end{aligned} \tag{8d}$$

where the second line in equation (8d) represents the effective resistance to flow through cased well–aquitard segments (the first term on the right-hand side) and well–aquifer segments (the second term on the right-hand side) that are connected between aquifers i and $i+k$. Equations (8a)–(8d) represent the coupling between leaky wells ($l = 1, \ldots, N_L$) and aquifers through the boundary conditions at the leaky wellbores. With respect to the upper and lower boundaries of the entire solution domain, the conditions at the top and bottom boundary of these leaky wells can be specified as either no-flow or zero hydraulic head buildup.

3. Development of Analytical Solutions

[22] After applying the Laplace transform to governing equations (1)–(8), we develop building blocks of the general solutions in a multilayered system. The building blocks constitute analytical solutions for (1) diffuse leakage induced by injection at a single injection well, (2) focused leakage through one leaky well, with one injection well and no diffuse leakage, and (3) combined diffuse and focused leakage with one injection well and one leaky well. Following each category of general solutions, particular solutions for a two-aquifer–one-aquitard system are used as examples to demonstrate the developed solution procedure and to compare with existing analytical solutions. We finally employ the superposition method to present the general solutions for a multilayered system consisting of N aquifers, N_I injection wells, and N_L leaky wells.

3.1. Diffuse Leakage Through Aquitards in a Multilayered System

[23] The first building block involves the analytical solutions for head buildup and leakage rates in a multilayered

system with one injection well and diffuse leakage. We simplify the notation by neglecting superscript "I" and subscript "m," since there is no need to distinguish between different injection wells. The solution procedure includes Laplace transform of the governing partial differential equations for both aquifers and aquitards shown in section 2, and solution of the resulting system of ordinary differential equations (ODEs) in the Laplace domain by eigenvalue analysis. Following the general solution for an N-aquifer system, an example of a two-aquifer–one-aquitard system is presented to demonstrate the solution procedure and compare it with existing analytical solutions.

3.1.1. Analytical Solution for an *N*-Aquifer System

[24] By applying the Laplace transform to equation (2a), for flow in aquitards, we obtain:

$$\frac{d^2\bar{s}_i^\alpha}{dz_{Di}^{\alpha\,2}} = \frac{B_i^{\alpha 2} p\,\bar{s}_i^\alpha}{D_i^\alpha}, \tag{9a}$$

where p is the Laplace transform variable, and the overbar denotes the Laplace transform of a variable. The Laplace transform of the boundary conditions for aquitard (i,α), equation (2b), becomes

$$\begin{aligned}\bar{s}_i^\alpha(r,\,0,\,p) &= \bar{s}_i(r,p),\\ \bar{s}_i^\alpha(r,\,1,\,p) &= \bar{s}_{i+\alpha}(r,p).\end{aligned} \tag{9b}$$

[25] The solution of equation (9a) subject to (9b) for the head buildup in aquitard (i,α) can be expressed [*Maas*, 1987b; *Cheng and Morohunfola*, 1993; *Zhou et al.*, 2009] as follows:

$$\bar{s}_i^\alpha(r,\,z_{Di}^\alpha,\,p) = \bar{s}_i\frac{\sinh\left[\kappa_i^\alpha\left(1-z_{Di}^\alpha\right)\right]}{\sinh\left(\kappa_i^\alpha\right)} + \bar{s}_{i+\alpha}\frac{\sinh\left(\kappa_i^\alpha z_{Di}^\alpha\right)}{\sinh\left(\kappa_i^\alpha\right)}, \tag{10a}$$

where

$$\kappa_i^\alpha(p) = \sqrt{p/D_i^\alpha}B_i^\alpha. \tag{10b}$$

[26] The Laplace transform of the flow equations for aquifers, equation (1a), leads to

$$\nabla_r^2\bar{s}_i = \frac{p}{D_i}\bar{s}_i + \frac{\bar{w}_i^-}{T_i} + \frac{\bar{w}_i^+}{T_i}. \tag{11a}$$

[27] After applying the Laplace transform to equation (1b), we obtain the transformed diffuse leakage, $\bar{w}_i^\alpha$:

$$\bar{w}_i^\alpha(r,p) = f_i^\alpha\,\bar{s}_i - g_i^\alpha\,\bar{s}_{i+\alpha} \tag{11b}$$

with

$$\begin{aligned}f_i^\alpha(p) &= \left(K_i^\alpha/B_i^\alpha\right)\kappa_i^\alpha\coth\left(\kappa_i^\alpha\right),\\ g_i^\alpha(p) &= \left(K_i^\alpha/B_i^\alpha\right)\kappa_i^\alpha\operatorname{csch}\left(\kappa_i^\alpha\right).\end{aligned} \tag{11c}$$

[28] By inserting equation (11b) into (11a), we obtain the following flow equations for all aquifers in the Laplace domain:

$$\nabla_r^2\bar{s}_i = -\frac{g_i^-}{T_i}\bar{s}_{i-1} + \left(\frac{p}{D_i} + \frac{f_i^+ + f_i^-}{T_i}\right)\bar{s}_i - \frac{g_i^+}{T_i}\bar{s}_{i+1};\;\; i = 1,\ldots,N. \tag{12a}$$

Equation (12a) is subject to the boundary condition at the injection well, transformed from equation (7):

$$-2\pi r_{wi}T_i\frac{d\bar{s}_i(r_{wi},p)}{dr} = \bar{Q}_i(p);\;\; i = 1,\ldots,N, \tag{12b}$$

and the condition at the outer lateral boundary, transformed from equation (4):

$$\bar{s}_i(r\to\infty,p) = 0. \tag{12c}$$

Equation (12a) forms a coupled system of ODEs and can be expressed in matrix notation:

$$\nabla_r^2\bar{\mathrm{s}} = \mathbf{A}\,\bar{\mathrm{s}}, \tag{13a}$$

where matrix $\mathbf{A}$, $[a_{i,j}]_{N\times N}$, referred to as the diffuse-leakage-coupling matrix, is a tri-diagonal matrix, with its components:

$$a_{i,i-1} = -\frac{g_i^-}{T_i},\;\; 2\le i\le N, \tag{13b}$$

$$a_{i,i} = \frac{p}{D_i} + \frac{f_i^+ + f_i^-}{T_i},\;\; 1\le i\le N, \tag{13c}$$

$$a_{i,i+1} = -\frac{g_i^+}{T_i},\;\; 1\le i\le N-1. \tag{13d}$$

[29] The system of ODEs, equation (13), can be decoupled by finding the eigenvalues λ and the eigenvectors ($\boldsymbol{\xi}$) for the following eigenvalue system [*Hunt*, 1985; *Hemker and Maas*, 1987; *Cheng and Morohunfola*, 1993]:

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\xi} = 0, \tag{14a}$$

where $\mathbf{I}$ is a unit diagonal matrix. For a general case, the transmisivity (T_i) of each aquifer may differ, and therefore, the diffuse-leakage-coupling matrix is an unsymmetrical $N\times N$ matrix. Following *Maas* [1986], matrix $\mathbf{A}$ can be transformed into a symmetrical matrix, $\mathbf{A}'$, using $\mathbf{A}' = \mathbf{T}^{1/2}\mathbf{A}\,\mathbf{T}^{-1/2}$ and $\boldsymbol{\xi}' = \mathbf{T}^{1/2}\boldsymbol{\xi}$ to guarantee that there exist N real positive eigenvalues associated with N real independent eigenvectors, where $\mathbf{T}$ is the diagonal transmissivity matrix with its components T_i. Then, equation (14a) can be rewritten as

$$(\mathbf{A}' - \lambda\mathbf{I})\boldsymbol{\xi}' = 0. \tag{14b}$$

[30] We use the eigenvalue analysis to solve equation (14b) and obtain the eigenvalues λ_i and the eigenvectors $\boldsymbol{\xi}'$, and thus $\boldsymbol{\xi} = \mathbf{T}^{-1/2}\boldsymbol{\xi}'$ that are normalized and orthonormal relative to $\mathbf{T}$ [*Hunt*, 1985].

[31] The fundamental solution for equation (13a) subject to the boundary condition in (12c) is expressed by $K_0\left(r\sqrt{\lambda_k}\right)$ [*Maas*, 1987b], where K_0 is the zeroth-order modified Bessel function of second kind. After employing the boundary condition at the wellbore, equation (12b), the analytical solution to equation (13a) can be written in matrix notation [*Hunt*, 1985; *Hemker and Maas*, 1987]:

$$\bar{\mathrm{s}} = \frac{1}{2\pi}\xi\,\mathbf{F}\xi^T\,\bar{Q} \tag{15a}$$

or

$$\bar{s}_i = \frac{1}{2\pi}\sum_{j=1}^{N}\sum_{k=1}^{N}\xi_{i,j}\frac{K_0\left(r\sqrt{\lambda_j}\right)}{E_{i,j}^I}\xi_{k,j}\bar{Q}_k;\quad i = 1,\ldots,N, \tag{15b}$$

where $\mathbf{F}$ is the diagonal matrix with its components $F_{j,j} = K_0(r\sqrt{\lambda_j})/E^I_{i,j}$, and $E^I_{i,j} = r_{wi}\sqrt{\lambda_j}K_1(r_{wi}\sqrt{\lambda_j})$ with $E^I_{i,j} = 1$ as $r_{wi} \to 0$, and K_1 is the first-order modified Bessel function of second kind. $\bar{Q}$ is the vector of transformed injection rates into each aquifer along the injection well. With computed eigenvalues and eigenvectors and known injection rates, equation (15b) can be evaluated very efficiently to obtain the Laplace-transformed head buildup in a multilayered system, caused by injection at a single well which may have multiple screen intervals. Equation (15b) can also be written in the form:

$$\bar{s}_i = \sum_{j=1}^{N} c^I_j\, \xi_{i,j} K_0\left(r\sqrt{\lambda_j}\right) \tag{15c}$$

with

$$c^I_j = 1/(2\pi E^I_{i,j}) \sum\nolimits_{k=1}^{N} \bar{Q}_k\, \xi_{k,j}, \tag{15d}$$

where c^I_j are the coefficients obtained from the boundary condition at the injection wellbore and expressed as a function of the Laplace variable p. Equations (15) are the generalized solutions for head buildup in a multilayered system caused by single-well fluid injection with diffuse leakage. They also form a general framework for developing an efficient solution method to the combined leakage problem in section 3.3.

[32] Note that the diffuse-leakage-coupling matrix, $\mathbf{A}$ in equation (13), depends only on the geometric and hydrogeologic properties of the aquifers and aquitards, as well as the Laplace variable p (representative of time), whereas it is independent of the radial distance (r) from the injection well. Therefore, the eigenvalue problem in equation (14b) is solved only once for a multilayered system at a given p, whether one or multiple injection (and leaky) wells may be involved. The eigenvalue system analysis is repeated for different p (or time t).

[33] The rate of diffuse leakage through the aquifer–aquitard interface between aquifer i and its neighboring aquitard (i, α) can be calculated by integrating equation (11b) over the entire interface area [*Zhou et al.*, 2009]:

$$\bar{Q}^\alpha_i = 2\pi \int_0^\infty \bar{w}^\alpha_i(r,p)r\,dr = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{\left(f^\alpha_i \xi_{i,j} - g^\alpha_i \xi_{i+\alpha,j}\right)}{\lambda_j} \xi_{k,j}\bar{Q}_k; \quad i = 1,\ldots,N. \tag{16}$$

[34] The total leakage rate depends on the hydraulic and geometric properties of both aquifers and aquitards, and on the injection rates.

[35] The derivation of the solutions discussed above is based on the boundary conditions of zero head buildup, equations (5a) and (6a), at the top and bottom of the entire system. For a system with a no-flow condition specified at the bottom boundary in equation (5b) (i.e., $i = 1$, $\alpha = -$), or at the top boundary in equation (6b) (i.e., $i = N$, $\alpha = +$), equations (10a) and (11c) for the corresponding aquitard(s) become [*Zhou et al.*, 2009]

$$\bar{s}^\alpha_i(r, z^\alpha_{Di}, p) = \bar{s}_i \frac{\cosh\left[\kappa^\alpha_i(1 - z^\alpha_{Di})\right]}{\cosh(\kappa^\alpha_i)}, \tag{17a}$$

$$f^\alpha_i(p) = \left(K^\alpha_i/B^\alpha_i\right)\kappa^\alpha_i \tanh\left(\kappa^\alpha_i\right) \qquad g^\alpha_i(p) = 0. \tag{17b}$$

[36] Correspondingly, the matrix coefficients $a_{1,1}$ and/or $a_{N,N}$ in the diffuse-leakage-coupling matrix need to be updated using equation (17b).

3.1.2. An Example for a Two-Aquifer–One-Aquitard System

[37] To demonstrate the solution procedure, we use, as an example, a system of two aquifers sandwiched by a leaky aquitard. A constant rate Q of injection into the bottom aquifer is used. For this two-aquifer–one-aquitard system, the diffuse-leakage-coupling matrix, $\mathbf{A}$, in equation (13), is

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} \dfrac{p}{D_1} + \dfrac{f'}{T_1} & -\dfrac{g'}{T_1} \\ -\dfrac{g'}{T_2} & \dfrac{p}{D_2} + \dfrac{f'}{T_2} \end{bmatrix}, \tag{18a}$$

where

$$f' = (K'/B')\kappa \coth(\kappa), \quad g' = (K'/B')\,\kappa\, \mathrm{csch}(\kappa), \quad \kappa = \sqrt{p/D'}B' \tag{18b}$$

and superscript ($'$) denotes the properties of the aquitard. The symmetrical transformed matrix is

$$\mathbf{A}' = \mathbf{T}^{1/2}\mathbf{A}\,\mathbf{T}^{-1/2} = \begin{bmatrix} a_{1,1} & \sqrt{T_1/T_2}\,a_{1,2} \\ \sqrt{T_2/T_1}\,a_{2,1} & a_{2,2} \end{bmatrix}. \tag{19a}$$

[38] The eigenvalues of $\mathbf{A}'$ are the roots of the characteristic equation $\det(\mathbf{A}' - \lambda\mathbf{I}) = 0$:

$$\lambda_1 = \frac{a_{1,1} + a_{2,2} + \sqrt{\left(a_{1,1} - a_{2,2}\right)^2 + 4a_{1,2}a_{2,1}}}{2}, \qquad \lambda_2 = \frac{a_{1,1} + a_{2,2} - \sqrt{\left(a_{1,1} - a_{2,2}\right)^2 + 4a_{1,2}a_{2,1}}}{2}. \tag{19b}$$

[39] The eigenvectors ($\boldsymbol{\xi}'$) corresponding to λ_1 and λ_2 are found from the solutions of the following homogeneous equations:

$$\begin{bmatrix} a_{1,1} - \lambda_1 & \sqrt{T_1/T_2}\,a_{1,2} \\ \sqrt{T_2/T_1}\,a_{2,1} & a_{2,2} - \lambda_1 \end{bmatrix} \begin{bmatrix} \xi'_{1,j} \\ \xi'_{2,j} \end{bmatrix} = 0; \quad j = 1, 2. \tag{20a}$$

[40] A pair of eigenvector values $\xi'_{i,j}$ for each eigenvalue λ_j (j = 1, 2) is normalized by dividing the square root of the sum of their squares. Then, computing $\boldsymbol{\xi} = \mathbf{T}^{-1/2}\boldsymbol{\xi}'$, we obtain the eigenvectors orthonormal with respect to T:

$$\begin{bmatrix} \xi_{1,1} & \xi_{1,2} \\ \xi_{2,1} & \xi_{2,2} \end{bmatrix} = \begin{bmatrix} \dfrac{a_{1,2}}{\sqrt{T_1 a^2_{1,2} + T_2\left(a_{1,1} - \lambda_1\right)^2}} & \dfrac{a_{1,2}}{\sqrt{T_1 a^2_{1,2} + T_2\left(a_{1,1} - \lambda_2\right)^2}} \\ \dfrac{-\left(a_{1,1} - \lambda_1\right)}{\sqrt{T_1 a^2_{1,2} + T_2\left(a_{1,1} - \lambda_1\right)^2}} & \dfrac{-\left(a_{1,1} - \lambda_2\right)}{\sqrt{T_1 a^2_{1,2} + T_2\left(a_{1,1} - \lambda_2\right)^2}} \end{bmatrix}. \tag{20b}$$

[41] With the computed eigenvalues and eigenvectors, the head-buildup solutions for the example are given by using equation (15b):

$$\bar{s}_1 = \frac{Q}{2\pi p}\left[\xi_{1,1}^2 K_0\left(r\sqrt{\lambda_1}\right)/E_{1,1}^I + \xi_{1,2}^2 K_0\left(r\sqrt{\lambda_2}\right)/E_{1,2}^I\right],$$
$$\bar{s}_2 = \frac{Q}{2\pi p}\left[\xi_{2,1}\xi_{1,1} K_0\left(r\sqrt{\lambda_1}\right)/E_{2,1}^I + \xi_{2,2}\xi_{1,2} K_0\left(r\sqrt{\lambda_2}\right)/E_{2,2}^I\right], \tag{21}$$

when $r_{wi} \to 0$, $E_{1,1}^I = E_{1,2}^I = E_{2,1}^I = E_{2,2}^I = 1$ and equation (21) is equivalent to equations (37a) and (37b) by *Cheng and Morohunfola* [1993] using a different solution methodology. The solution technique discussed above is more efficient for a multilayered system with $N > 2$ [*Cheng*, 1994; *Hemker and Maas*, 1994].

3.2. Focused Leakage Through a Well in a Multilayered System

[42] This section describes the second building block for focused leakage through a leaky well, with no diffuse leakage (i.e., assuming impervious aquitards). Flow in the leaky well is driven by hydraulic head buildup in the multilayered system caused by injection at an injection well. In this case, the head buildup in different aquifers is coupled only by the well leakage.

3.2.1. Analytical Solution for an *N*-Aquifer System

[43] We first transform the boundary condition (8a) with the associated continuity equation in equation (8c) for leakage rates through a leaky well ($l = 1$) and obtain

$$-2\pi r_{Li} T_i \frac{\partial \bar{s}_i^L(r_{Li}, p)}{\partial r} = \bar{u}_i(p), \tag{22a}$$

$$\bar{u}_i(p) = \frac{1}{\Omega_i^-}\left(\bar{s}_{i-1,l}^T - \bar{s}_{i,l}^T\right) - \frac{1}{\Omega_i^+}\left(\bar{s}_{i,l}^T - \bar{s}_{i+1,l}^T\right), \tag{22b}$$

where the total head buildup evaluated at the leaky well is $\bar{s}_{i,l}^T = \bar{s}_{i,l}^I + \bar{s}_{i,l}^L$. Superscripts I and L are used to denote the head buildup corresponding to the injection well and the leaky well, respectively. The second subscript l indicates that the head buildups are evaluated at the leaky well.

[44] In the case of impervious aquitards, the head buildup in aquifer i caused by injection can be obtained by solving equation (12a) with $f_i^\alpha = g_i^\alpha = 0$, subject to equations (12b) and (12c), and expressed as

$$\bar{s}_i^I = \frac{\bar{Q}_i}{2\pi T_i E_i^I} K_0\left(r^I\sqrt{p/D_i}\right), \tag{23}$$

where $E_i^I = r_{wi}\sqrt{p/D_i}K_1(r_{wi}\sqrt{p/D_i})$. Note that in contrast to the previous section, E_i has one subscript here since the eigenvalues depend only on the properties of the injection aquifers. r^I is the radial distance from the injection well. In the case of a negligibly small well radius ($r_{wi} \to 0$) and using a constant injection rate (Q_i), we have $\bar{s}_i^I = Q_i/(2\pi T_i p)K_0(r^I\sqrt{p/D_i})$, which is the Theis solution in the Laplace domain [*Theis*, 1935; *Zhou et al.*, 2009].

[45] Similarly, the Laplace-transformed head buildup, $\bar{s}_i^L$, in aquifer i caused by well leakage at the leaky well can be expressed in terms of the leakage rate $\bar{u}_i$,

$$\bar{s}_i^L = \frac{\bar{u}_i}{2\pi T_i E_i^L} K_0\left(r^L\sqrt{p/D_i}\right), \tag{24a}$$

where $E_i^L = r_{Li}\sqrt{p/D_i}K_1(r_{Li}\sqrt{p/D_i})$, r^L is the radial distance from the leaky well, and r_{Li} is the radius of the leaky well–aquifer i segment. By superposition, the total head buildup at an observation point located at r^I and r^L distances away from the injection and leaky wells, respectively, is given by

$$\bar{s}_i^T = \frac{\bar{Q}_i}{2\pi T_i E_i^I} K_0\left(r^I\sqrt{p/D_i}\right) + \frac{\bar{u}_i}{2\pi T_i E_i^L} K_0\left(r^L\sqrt{p/D_i}\right). \tag{24b}$$

[46] Substituting the total head buildup, equation (24b), into the continuity equation on the right-hand side of equation (22b), we obtain a linear system of equations in terms of the unknown leakage rates in well–aquifer segments of the leaky well,

$$b_{i,i-1}\bar{u}_{i-1} + b_{i,i}\bar{u}_i + b_{i,i+1}\bar{u}_{i+1} = d_{i,i-1}\bar{s}_{i-1,l}^I + d_{i,i}\bar{s}_{i,l}^I + d_{i,i+1}\bar{s}_{i+1,l}^I \tag{25}$$

with

$$\begin{aligned}
b_{i,i-1} &= \frac{-1}{2\pi T_{i-1}E_{i-1}^L\Omega_i^-} K_0\left(r_{Li-1}\sqrt{p/D_{i-1}}\right),\\
b_{i,i} &= 1 + \frac{1}{2\pi T_i E_i^L}\left(\frac{\Omega_i^+ + \Omega_i^-}{\Omega_i^+\Omega_i^-}\right) K_0\left(r_{Li}\sqrt{p/D_i}\right),\\
b_{i,i+1} &= \frac{-1}{2\pi T_{i+1}E_{i+1}^L\Omega_i^+} K_0\left(r_{Li+1}\sqrt{p/D_{i+1}}\right),\\
d_{i,i-1} &= \frac{1}{\Omega_i^-},\ d_{i,i} = -\frac{\Omega_i^+ + \Omega_i^-}{\Omega_i^+\Omega_i^-},\ d_{i,i+1} = \frac{1}{\Omega_i^+},\\
\bar{s}_{i,l}^I &= \frac{\bar{Q}_i}{2\pi T_i E_i^I} K_0\left(R\sqrt{p/D_i}\right),
\end{aligned} \tag{26}$$

where R is the distance from the injection well to the leaky well. We refer to the resulting matrix, **B**, as the focused-leakage-coupling matrix, which represents the coupling by leakage through the leaky well(s). In the case of a single leaky well, matrix **B** (of $N \times N$) is a tri-diagonal matrix. If multiple leaky wells are involved, the contributions of the additional leaky wells to head buildup are added to equation (24b) and matrix **B** in equation (25) is no longer tri-diagonal. For N aquifers and N_L leaky wells, a system of $N \times N_L$ linear algebraic equations can be solved using a standard matrix-vector solver for the $N \times N_L$ unknown leakage rates $\bar{u}_i$. Then the head buildup in each aquifer can be obtained by superposition of the solutions for all the wells.

3.2.2. An Example for a Two-Aquifer–One-Aquitard System

[47] To demonstrate the solution procedure and compare with an existing analytical solution, we again use the two-aquifer–one-aquitard system, but with one injection well, one leaky well, and no diffuse leakage. A constant rate (Q) is used for injection into the bottom aquifer. For this system, equation (24b) can be written for both aquifers 1 and 2 as

$$\begin{aligned}
\bar{s}_1^T &= \frac{Q}{2\pi T_1 p E_1^I} K_0\left(r^I\sqrt{p/D_1}\right) + \frac{\bar{u}_1}{2\pi T_1 E_1^L} K_0\left(r^L\sqrt{p/D_1}\right)\\
\bar{s}_2^T &= \frac{\bar{u}_2}{2\pi T_2 E_2^L} K_0\left(r^L\sqrt{p/D_2}\right).
\end{aligned} \tag{27}$$

[48] By using equation (25), we obtain two equations with two unknowns:

$$
\begin{bmatrix} 1+1/(2\pi T_1 E_1^L \Omega')K_0(r_{L1}\sqrt{p/D_1}) & -1/(2\pi T_2 E_2^L \Omega')K_0(r_{L2}\sqrt{p/D_2}) \\ -1/(2\pi T_1 E_1^L \Omega')K_0(r_{L1}\sqrt{p/D_1}) & 1+1/(2\pi T_2 E_2^L \Omega')K_0(r_{L2}\sqrt{p/D_2}) \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} = \begin{Bmatrix} -Q\, K_0(R\sqrt{p/D_1})/(2\pi T_1 E_1^I p\, \Omega') \\ Q\, K_0(R\sqrt{p/D_1})/(2\pi T_1 E_1^I p\, \Omega') \end{Bmatrix}. \quad (28)
$$

[49] The solution for the leakage rate in the Laplace domain is

$$
\bar{u}_1 = -\bar{u}_2 = \frac{-Q/(2\pi T_1 E_1^I) K_0\left(R\sqrt{p/D_1}\right)}{p\left[\Omega' + K_0\left(r_{L1}\sqrt{p/D_1}\right)/\left(2\pi T_1 E_1^L\right) + K_0\left(r_{L2}\sqrt{p/D_2}\right)/\left(2\pi T_2 E_2^L\right)\right]}. \quad (29)
$$

[50] The total head buildup in the entire aquifer system can be obtained using equation (27). When the radii of both the injection and the leaky wells are negligibly small, i.e., $E_1^I = E_1^L = E_2^L = 1$, equations (27) and (29) become identical to equations (7) and (12) presented in the work of *Avci* [1994]. Note that we assumed an initial hydrostatic condition, while *Avci* [1994] presented his solutions by assuming an initially uniform ambient hydraulic gradient between the two aquifers. Existence of initial uniform head differences in the aquifers can be incorporated into equation (29), without complicating the solution by any means for the case of focused leakage only. If the resistance in the leaky well approaches zero such as in the case of an open borehole, the maximum possible leakage rate into the overlying aquifer can be estimated by substitution of $\Omega' = 0$ in equation (29).

3.3. Coupled Diffuse and Focused Leakage in a Multilayered System

[51] The third building block for coupled diffuse and focused leakage in an N-aquifer system is presented in this section, starting with one injection well and one leaky well. The solution procedure combines the solutions presented in sections 3.1 and 3.2 for diffuse leakage and for focused leakage, respectively. This building block is a generalized solution; it will be extended to a system of multiple injection wells and multiple leaky wells in section 3.4.

3.3.1. Analytical Solution for an *N*-Aquifer System

[52] Equation (15b) is used for calculating the head buildup caused by the injection at an injection well in the presence of diffuse leakage. Similarly, the head buildup caused by leakage through a leaky well can be expressed, using equation (15c), as

$$
\bar{s}_i^L = \sum_{j=1}^{N} c_j^L\, \xi_{i,j} K_0\left(r^L \sqrt{\lambda_j}\right); \quad i = 1, \ldots, N. \quad (30a)
$$

[53] Using the superposition principle, the total head buildup caused by both injection and leakage is given by

$$
\bar{s}_i^T = \bar{s}_i^I + \bar{s}_i^L. \quad (30b)
$$

[54] Note that the flow rate at a well–aquifer segment may be positive (for recharge from the leaky well) or negative (for leakage into the leaky well). Also, the hydraulic head perturbation at well–aquifer segments can be positive or negative; however, we use the term "head buildup" here in both cases. The eigenvalues and eigenvectors as described in section 3.1 are the same for calculating injection- or leakage-induced head buildup. The coefficients c_j^L must be obtained from the boundary conditions at the leaky well that were presented in equation (22). By taking a derivative of equation (30a) and then substituting it into equation (22), we obtain a system of algebraic equations in the form of:

$$
\begin{aligned}
&\sum_{j=1}^{N} c_j^L\, b_{1,j} = -\frac{1}{\Omega_1^+}\bar{s}_{1,l}^I + \frac{1}{\Omega_1^+}\bar{s}_{2,l}^I;\ i = 1,\\
&\sum_{j=1}^{N} c_j^L\, b_{i,j} = \frac{1}{\Omega_i^-}\bar{s}_{i-1,l}^I - \frac{\Omega_i^+ + \Omega_i^-}{\Omega_i^+\Omega_i^-}\bar{s}_{i,l}^I + \frac{1}{\Omega_i^+}\bar{s}_{i+1,l}^I;\ 2 \le i \le N-1,\\
&\sum_{j=1}^{N} c_j^L\, b_{N,j} = \frac{1}{\Omega_N^-}\bar{s}_{N-1,l}^I - \frac{1}{\Omega_N^-}\bar{s}_{N,l}^I;\ i = N,
\end{aligned} \quad (31a)
$$

with

$$
\begin{aligned}
b_{1,j} &= \left[2\pi K_1 B_1\, r_{L1}\sqrt{\lambda_j} K_1\left(r_{L1}\sqrt{\lambda_j}\right) + \frac{1}{\Omega_1^+} K_0\left(r_{L1}\sqrt{\lambda_j}\right)\right]\xi_{1,j} - \frac{1}{\Omega_1^+}K_0\left(r_{L2}\sqrt{\lambda_j}\right)\xi_{2,j},\\
b_{i,j} &= -K_0\left(r_{Li-1}\sqrt{\lambda_j}\right)\frac{1}{\Omega_i^-}\xi_{i-1,j} + \left[2\pi K_i B_i\, r_{Li}\sqrt{\lambda_j} K_1\left(r_{Li}\sqrt{\lambda_j}\right) + \frac{\Omega_i^+ + \Omega_i^-}{\Omega_i^+\Omega_i^-} K_0\left(r_{Li}\sqrt{\lambda_j}\right)\right]\xi_{i,j} - \frac{1}{\Omega_i^+}K_0\left(r_{Li+1}\sqrt{\lambda_j}\right)\xi_{i+1,j};\\
&\quad 2 \le i \le N-1,\\
b_{N,j} &= -K_0\left(r_{LN-1}\sqrt{\lambda_j}\right)\frac{1}{\Omega_N^-}\xi_{N-1,j} + \left[2\pi K_N B_N\, r_{LN}\sqrt{\lambda_j} K_1\left(r_{LN}\sqrt{\lambda_j}\right) + \frac{1}{\Omega_N^-}K_0\left(r_{LN}\sqrt{\lambda_j}\right)\right]\xi_{N,j},
\end{aligned} \quad (31b)
$$

where equations (31a) and (31b) were written by assuming the top and bottom of the leaky well are closed (no-flow boundaries). These equations can be modified for other types of boundary conditions. For instance, zero buildup at the top of the well can be assigned for a flowing or an artesian well, in which case the last line in equation (31a) is modified for $i = N$: $\sum_{j=1}^{N} c_j^L\, b_{N,j} = (1/\Omega_N^-)\,\bar{s}_{N-1,l}^I - (\Omega_N^- + \Omega_N^+)/(\Omega_N^-\,\Omega_N^+)\,\bar{s}_{N,l}^I$.

[55] The focused-leakage-coupling matrix in equation (31) for the combined leakage problem forms an $N \times N$ matrix for the single leaky well. The system of linear equations is solved for the unknown c_j^L at each p (or time t) by using linear matrix-solver routines. The leakage rate at a leaky well–aquifer segment in the Laplace domain can be expressed by substituting equation (30a) into (22a):

$$\bar{u}_i = 2\pi T_i \sum_{j=1}^{N} c_j^L \xi_{i,j} E_{i,j}^L, \tag{31c}$$

where $E_{i,j}^L = r_{Li}\sqrt{\lambda_j} K_1(r_{Li}\sqrt{\lambda_j})$. Similar to equation (16), the rate of diffuse leakage from aquifer i to aquitard (i, α) can be calculated using

$$\bar{Q}_i^\alpha = \sum_{j=1}^{N} \left\{ \frac{(f_i^\alpha \xi_{i,j} - g_i^\alpha \xi_{i+\alpha,j})}{\lambda_j} \left[\left(\sum_{k=1}^{N} \xi_{k,j} \bar{Q}_k \right) + c_j^L \right] \right\}. \tag{32}$$

3.3.2. An Example for a Two-Aquifer–One-Aquitard System

[56] We again use the two-aquifer–one-aquitard system with one injection well and one leaky well presented in section 3.2.2. The only difference is that the aquitard in this example is permeable and allows diffuse leakage into and through it. For this example, the total head buildup in aquifers 1 and 2 is expressed as,

$$\begin{aligned} \bar{s}_1^T &= \frac{Q}{2\pi p}\left[\xi_{1,1}^2/E_{1,1}^I K_0\left(r^I\sqrt{\lambda_1}\right) + \xi_{1,2}^2/E_{1,2}^I K_0\left(r^I\sqrt{\lambda_2}\right)\right] \\ &\quad + \left[c_1^L \xi_{1,1} K_0\left(r^L\sqrt{\lambda_1}\right) + c_2^L \xi_{1,2} K_0\left(r^L\sqrt{\lambda_2}\right)\right], \\ \bar{s}_2^T &= \frac{Q}{2\pi p}\left[\xi_{1,1}\xi_{2,1}/E_{2,1}^I K_0\left(r^I\sqrt{\lambda_1}\right) + \xi_{1,2}\xi_{2,2}/E_{2,2}^I K_0\left(r^I\sqrt{\lambda_2}\right)\right] \\ &\quad + \left[c_1^L \xi_{2,1} K_0\left(r^L\sqrt{\lambda_1}\right) + c_2^L \xi_{2,2} K_0\left(r^L\sqrt{\lambda_2}\right)\right]. \end{aligned} \tag{33}$$

[57] The eigenvalues λ_i and the eigenvectors $\xi_{i,j}$ for the two-aquifer–one-aquitard system are given in equations (19b) and (20b). The coefficients c_1^L and c_2^L must be obtained from the boundary condition at the wellbore by employing equation (22):

$$\begin{aligned} c_1^L &= -\frac{\left(\bar{s}_{1,l}^I - \bar{s}_{2,l}^I\right)}{\Omega'} \frac{(b_{1,2} + b_{2,2})}{(b_{1,1}b_{2,2} - b_{1,2}b_{2,1})}, \\ c_2^L &= -c_1^L \frac{(b_{1,1} + b_{2,1})}{(b_{1,2} + b_{2,2})}, \end{aligned} \tag{34a}$$

where

$$\begin{aligned} b_{1,1} &= \left[2\pi T_1 E_{1,1}^L + \frac{1}{\Omega'} K_0\left(r_{L1}\sqrt{\lambda_1}\right)\right]\xi_{1,1} - \frac{1}{\Omega'} K_0\left(r_{L2}\sqrt{\lambda_1}\right)\xi_{2,1}, \\ b_{1,2} &= \left[2\pi T_1 E_{1,2}^L + \frac{1}{\Omega'} K_0\left(r_{L1}\sqrt{\lambda_2}\right)\right]\xi_{1,2} - \frac{1}{\Omega'} K_0\left(r_{L2}\sqrt{\lambda_2}\right)\xi_{2,2}, \\ b_{2,1} &= -\frac{1}{\Omega'} K_0\left(r_{L1}\sqrt{\lambda_1}\right)\xi_{1,1} + \left[2\pi T_2 E_{2,1}^L + \frac{1}{\Omega'} K_0\left(r_{L2}\sqrt{\lambda_1}\right)\right]\xi_{2,1}, \\ b_{2,2} &= -\frac{1}{\Omega'} K_0\left(r_{L1}\sqrt{\lambda_2}\right)\xi_{1,2} + \left[2\pi T_2 E_{2,2}^L + \frac{1}{\Omega'} K_0\left(r_{L2}\sqrt{\lambda_2}\right)\right]\xi_{2,2}, \end{aligned}$$

$$\begin{aligned} \bar{s}_{1,l}^I - \bar{s}_{2,l}^I = \frac{Q}{2\pi p}\Bigg[& \xi_{1,1}\left(\frac{\xi_{1,1}}{E_{1,1}^I} - \frac{\xi_{2,1}}{E_{2,1}^I}\right) K_0\left(R\sqrt{\lambda_1}\right) \\ & + \xi_{1,2}\left(\frac{\xi_{1,2}}{E_{1,2}^I} - \frac{\xi_{2,2}}{E_{2,2}^I}\right) K_0\left(R\sqrt{\lambda_2}\right)\Bigg]. \end{aligned} \tag{34b}$$

[58] When the radii of the leaky well at aquifer segments 1 and 2 are the same, i.e., $r_{L1} = r_{L2}$ and $E_{1,1}^L = E_{2,1}^L, E_{1,2}^L = E_{2,2}^L$, the coefficients c_1^L and c_2^L simplify to

$$\begin{aligned} c_1^L = &-\left(\bar{s}_{1,l}^I - \bar{s}_{2,l}^I\right)\left(\xi_{1,2}T_1/T_2 + \xi_{2,2}\right)\Big[\Omega' 2\pi T_1 E_{1,1}^L\left(\xi_{1,1}\xi_{2,2} - \xi_{1,2}\xi_{2,1}\right) \\ &+ K_0\left(r_{L1}\sqrt{\lambda_1}\right)\left(\xi_{1,2}T_1/T_2 + \xi_{2,2}\right)\left(\xi_{1,1} - \xi_{2,1}\right) \\ &+ K_0\left(r_{L1}\sqrt{\lambda_2}\right)\left(\xi_{1,1}T_1/T_2 + \xi_{2,1}\right)E_{1,1}^L/E_{1,2}^L\left(\xi_{2,2} - \xi_{1,2}\right)\Big]^{-1}, \\ c_2^L = &-c_1^L\left(\frac{\xi_{1,1}T_1/T_2 + \xi_{2,1}}{\xi_{1,2}T_1/T_2 + \xi_{2,2}}\right)\frac{E_{1,1}^L}{E_{1,2}^L}, \end{aligned} \tag{34c}$$

then, the leakage rate into or from the aquifers through the leaky well can be calculated using equation (31c):

$$\begin{aligned} \bar{u}_1 = -\bar{u}_2 = &-\left(\bar{s}_{1,l}^I - \bar{s}_{2,l}^I\right)\left(\xi_{1,1}\xi_{2,2} - \xi_{1,2}\xi_{2,1}\right) \\ &\Big[\Omega'\left(\xi_{1,1}\xi_{2,2} - \xi_{1,2}\xi_{2,1}\right) + 1/(2\pi E_{1,1}^L)K_0\left(r_{L1}\sqrt{\lambda_1}\right) \\ &\left(\xi_{1,2}/T_2 + \xi_{2,2}/T_1\right)\left(\xi_{1,1} - \xi_{2,1}\right) \\ &+ 1/(2\pi E_{1,2}^L)K_0\left(r_{L1}\sqrt{\lambda_2}\right) \\ &\left(\xi_{1,1}/T_2 + \xi_{2,1}/T_1\right)\left(\xi_{2,2} - \xi_{1,2}\right)\Big]^{-1}. \end{aligned} \tag{35}$$

[59] When the aquitards are impervious, i.e., $\bar{s}_{2,l}^I = \xi_{1,2} = \xi_{2,1} = 0$, $\xi_{1,1} = 1/\sqrt{T_1}$, $\xi_{2,2} = 1/\sqrt{T_2}$, $\lambda_1 = \sqrt{p/D_1}$ and $\lambda_2 = \sqrt{p/D_2}$, equation (35) reduces to equation (29) with $r_{L1} = r_{L2}$.

3.4. Generalized Solutions for Multiple Active and Leaky Wells

[60] The solutions obtained in section 3.3 can be generalized to a system of N aquifers, N_I injection wells, and N_L leaky wells. By using the superposition principle, the total head buildup in aquifer i is given as

$$\begin{aligned} \bar{s}_i^T = &\frac{1}{2\pi}\sum_{m=1}^{N_I}\sum_{j=1}^{N}\sum_{k=1}^{N}\xi_{i,j}\frac{K_0\left(r_m^I\sqrt{\lambda_j}\right)}{E_{i,j,m}^I}\xi_{k,j}\bar{Q}_{k,m} \\ &+ \sum_{l=1}^{N_L}\sum_{j=1}^{N} c_{j,l}^L \xi_{i,j} K_0\left(r_l^L\sqrt{\lambda_j}\right), \end{aligned} \tag{36}$$

where r_m^I is the distance between an observation point at the global horizontal coordinate (x, y) and injection well m located at (x_m^I, y_m^I), r_l^L is the distance between the observation point and leaky well l at (x_l^L, y_l^L), and $E_{i,j,m}^I = r_{wi,m}\sqrt{\lambda_j}K_1(r_{wi,m}\sqrt{\lambda_j})$. For each injection or leaky well, the distances from the well center can be calculated using

$$\begin{aligned} r_m^I &= \sqrt{\left(x - x_m^I\right)^2 + \left(y - y_m^I\right)^2}, \\ r_l^L &= \sqrt{\left(x - x_l^L\right)^2 + \left(y - y_l^L\right)^2}. \end{aligned} \tag{37}$$

[61] Note that the eigenvalues and the eigenvectors need to be calculated only once for a given p, no matter how many injection and leaky wells are involved. The system of $N \times N_L$ linear algebraic equations based on equation (31a) is solved using a standard matrix-vector solver for calculating the unknown $N \times N_L$ coefficients $c^L_{j,l}$ for N_L leaky wells ($l = 1, \ldots, N_L$). The total leakage rates into or from aquifers through the N_L leaky wells can be calculated as

$$\bar{u}_i = 2\pi T_i \sum_{l=1}^{N_L} \sum_{j=1}^{N} c^L_{j,l}\, \xi_{i,j} E^L_{i,j,l}, \tag{38}$$

where $E^L_{i,j,l} = r_{Li,l}\sqrt{\lambda_j} K_1\left(r_{Li,l}\sqrt{\lambda_j}\right)$. The total diffuse leakage rate through the aquitard–aquifer interfaces can be calculated using

$$\bar{Q}^\alpha_i = \sum_{j=1}^{N} \left\{ \frac{\left(f^\alpha_i \xi_{i,j} - g^\alpha_i \xi_{i+\alpha,j}\right)}{\lambda_j} \left[\left(\sum_{m=1}^{N_I} \sum_{k=1}^{N} \xi_{k,j} \bar{Q}_{k,m} \right) + \sum_{l=1}^{N_L} c^L_{j,l} \right] \right\}. \tag{39}$$

4. Solution Verification

[62] The generalized analytical solutions presented in section 3 were verified, using the solution procedure presented in the appendix, in comparison with existing analytical solutions and (if these were not available) with numerical simulation results. The first verification tests focused on leakage in a two-aquifer–one-aquitard system using the exact analytical solution of *Avci* [1994] for one leaky well, as well as the approximate analytical solution of *Nordbotten et al.* [2004] for a varying number of leaky wells. Verification for diffuse leakage was conducted against the exact analytical solution of *Hemker and Maas* [1987] and *Maas* [1987b] for a system of two aquifers and three leaky aquitards. The final verification case involved coupled diffuse and focused leakage in comparison with results from a numerical simulation. For this purpose, we used the same two-aquifer–three-aquitard system as described by *Maas* [1987b], but added one leaky well.

4.1. Solution Verification for Leakage Through Leaky Well(s)

[63] We evaluate here a two-aquifer–one-aquitard system with one injection well and one leaky well, as previously considered in the exact analytical solution by *Avci* [1994]. A mathematical verification for this case was already presented in section 3.2.2, i.e., it was shown that our new generalized solution applied to this specific example reduces to exactly the same form as in the existing solution. For additional numerical verification, we compare here the leakage rate calculated using our solution procedure to the results given in Table 1 of *Avci* [1994]. For comparison, the dimensionless radius of the leaky well, the leaky well resistance, and the hydraulic conductivity ratio between the upper and the lower aquifer were chosen from Table 1 in the work of *Avci* [1994] as $r'_a = 0.001$, $\Omega' = 0, 100$, and $\alpha = 0.1, 10$, respectively. To arrive at the same (or similar) values for these dimensionless parameters, we used typical hydrogeologic and geometric parameters for the aquifers and the leaky well as follows: The upper and lower aquifers are each 20 m thick, separated by an impervious aquitard of 10 m thickness. The lower aquifer has a hydraulic conductivity of $K_1 = 10^{-6}$ m s^{-1} and a specific storativity of 10^{-5} m^{-1}. The upper aquifer has the same specific storativity but different hydraulic conductivity value, which was chosen such that the ratios K_2/K_1 (i.e., α in the work of *Avci* [1994]) become 10 and 0.1. The leaky well has a radius of 0.2 m, and a hydraulic conductivity of 1.0 m s^{-1} or 10^{-4} m s^{-1}. The injection well is located 200 m away from the leaky well, with a constant unit rate of injection into the lower aquifer. Note that one of the test cases by *Avci* [1994] has a leaky well resistance of $\Omega' = 0$, which essentially represents an open-wellbore case with an infinitely high well conductivity. We simulated this case by using a very high (but finite) conductivity of 1.0 m s^{-1} in our code, corresponding to a leaky well resistance of $\Omega' = 0.01$ whose effect is practically equivalent to that of zero resistance ($\Omega' = 0$).

[64] Figure 2a exhibits excellent agreement between the normalized leakage rates (given as a fraction of the unit injection rate) calculated using our analytical solution and the values presented in Table 1 of *Avci* [1994]. Of the three selected cases shown, the leakage rates are highest for the case with a well conductivity of $K_L = 1$ m s^{-1} and $K_2/K_1 = 10$. Significantly less leakage occurs as (1) the leaky pathway is less permeable ($K_L = 10^{-4}$), or (2) the upper aquifer is less permeable ($K_2/K_1 = 0.1$). In the latter, leakage will result in greater pressure buildup locally around the leaky well in the upper aquifer and thus a lower vertical hydraulic gradient (driving force) along the leaky well. Thus, the reduced driving force results in less leakage from the bottom aquifer to the top aquifer. Figure 2b shows the transient hydraulic head buildup at an observation well 200 m away from both the injection and leaky wells, for the high-leakage case with $K_2/K_1 = 10$ and $K_L = 1$ m s^{-1}. Again, we observe excellent agreement between the analytical solutions for the head buildup in both aquifers. Because of the leakage effect, the late-time head buildup (s_1) in the injection aquifer deviates from (becomes less than) the logarithmic curve represented by the Theis solution.

[65] To demonstrate our analytical solutions for problems involving a larger number of hydrogeologic layers, we evaluated additional leakage cases with, respectively, two, three, and four aquifers and alternating aquitards overlying the injection aquifer. The overlying aquifers have the same parameters as the injection aquifer, and all aquitards have identical properties. The leaky well, which is screened within each aquifer, has a conductivity of $K_L = 1$ m s^{-1}. Figure 3 shows the normalized leakage rate for (1) flow from the injection aquifer into the leaky well, and (2) flow from the leaky well into the top aquifer. We note that the leakage rate *out of* the injection aquifer increases with the number of aquifers (because the driving force for leakage within the well increases), while the leakage rate *into* the top aquifer decreases (as an increasing fraction of the leaking fluid in the well recharges laterally into the overlying aquifers). *Nordbotten et al.* [2004] referred to this multilayered system behavior as an "elevator effect."

[66] A scenario with multiple leaky wells is considered in our second verification example, which involves comparison against the approximate analytical solution given by *Nordbotten et al.* [2004]. The test problem consists of

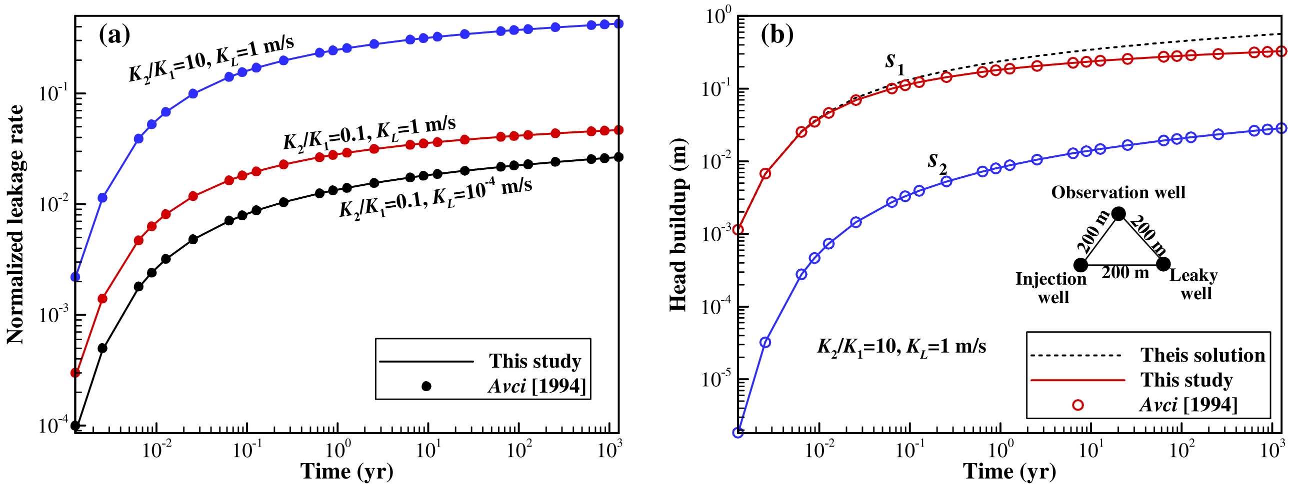

Figure 2. Verification of the developed analytical solutions for a two-aquifer–one-aquitard system with focused leakage only and a single leaky well, by comparing (a) the calculated transient normalized rate of well leakage and (b) the calculated head buildup in both aquifers with their existing solutions given by *Avci* [1994].

two aquifers with identical hydraulic conductivity (i.e., 2×10^{-7} m s^{-1}) and identical storage coefficient (i.e., $S_s \times B = 5 \times 10^{-7}$), bounded at the top and the bottom by impervious boundaries. The lower and upper aquifers are 20 and 30 m thick, respectively, and the separating impermeable aquitard is 15 m thick. The system also contains one injection well operating at a constant injection rate and a varying number ($N_L = 1$ to 10) of leaky wells arranged in a regular circular pattern at 1 m distance from the injection well [*Nordbotten*, personal communication, 2011]. The leaky wells all have a radius of 0.15 m and a hydraulic conductivity of 2×10^{-4} m s^{-1}.

[67] Figure 4 shows the total well-leakage rate (normalized by the injection rate) as a function of the number of leaky wells. Our exact solution is in a reasonable agreement with the approximate solution of *Nordbotten et al.* [2004], and in an excellent agreement with a high-resolution numerical simulation conducted with the COMSOL multiphysics package. We may postulate that the minor underestimation of leakage rate by *Nordbotten et al.*'s [2004] results is caused by approximations in their solution procedure: In their solution, the convolution integral in the real time domain is approximated (1) by using only the first few finite terms of the infinite-series well function, and (2) by replacing the time-varying leakage rate in leaky wells with a Heaviside step function.

4.2. Solution Verification for Diffuse Leakage

[68] We utilize here the exact analytical solution for diffuse leakage in a two-aquifer–three-aquitard system by *Maas* [1987b] and *Hemker and Maas* [1987]. The aquifer system consists of two aquifers and three alternating aquitards of

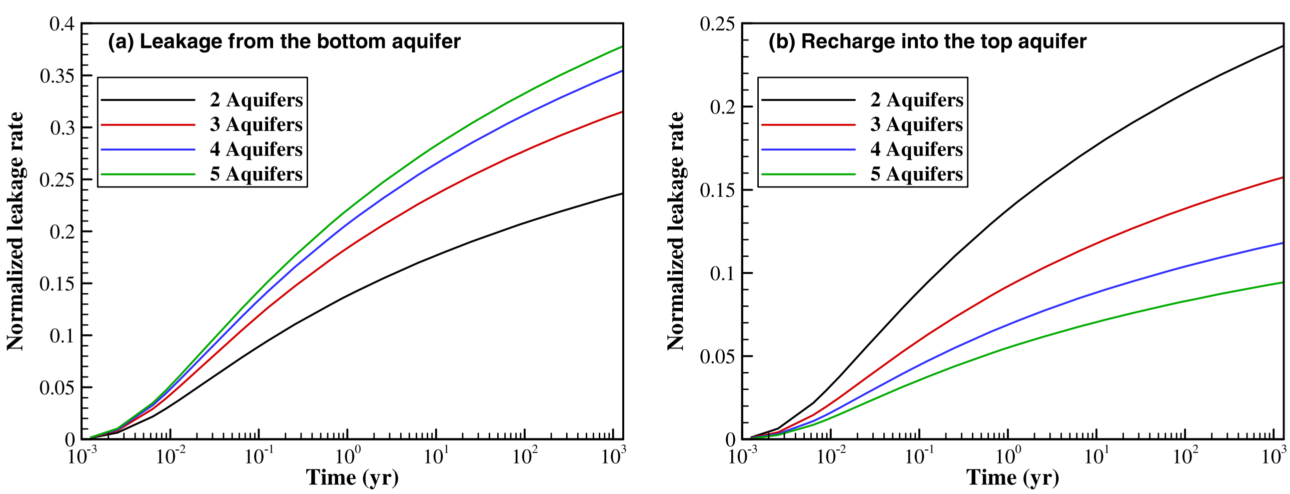

Figure 3. Normalized rate of well leakage in multilayered systems with varying number of aquifers and alternating aquitards (a) from the bottom (injection) aquifer and (b) into the top aquifer calculated using the developed analytical solutions for multilayered systems, showing the elevator effect of successive well leakage.

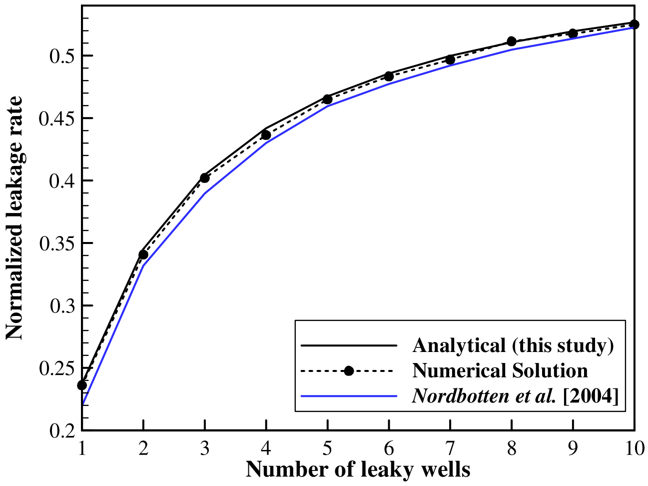

Figure 4. Verification of the developed analytical solutions for a two-aquifer–one-aquitard system with focused leakage only and varying number of leaky wells, by comparing the calculated normalized leakage rate, as a function of the number of leaky wells involved, with the existing approximate solution by *Nordbotten et al.* [2004] and a high-resolution numerical solution.

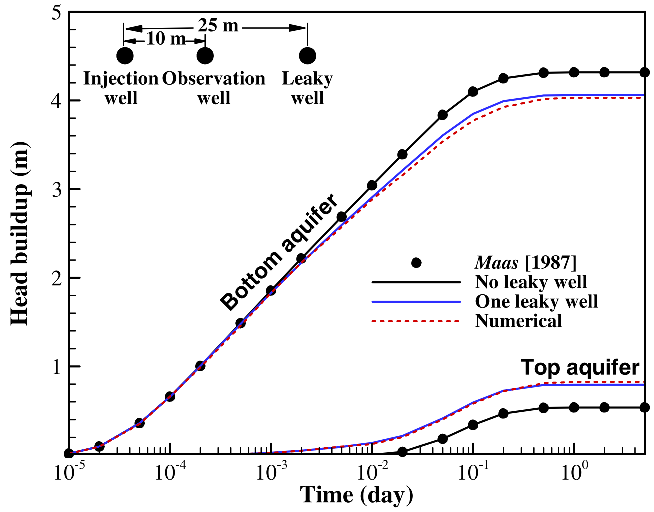

Figure 5. Verification of the developed analytical solutions for a two-aquifer–three-aquitard system by comparing the calculated transient head buildup at the observation well in both aquifers with the analytical solution by *Maas* [1987b] and *Hemker and Maas* [1987] in the case of diffuse leakage only, and with a numerical solution in the case of coupled diffuse and focused leakage with one leaky well.

infinite lateral extension, with a condition of zero pressure change at the top and bottom boundaries. The two aquifers have the same transmissivity of $T = 100$ m^2 d^{-1}, and an identical storage coefficient $S_s \times B = 10^{-4}$. The aquitards have the same thickness and have hydraulic conductivity and specific storativity values assigned such that $B'/K' = 100$ d and $S_s' \times B' = 1.6 \times 10^{-3}$. Injection occurs into the bottom aquifer at a constant rate of 400 π m^3 d^{-1}. An observation well is located 10 m away from the injection well.

[69] Figure 5 shows the hydraulic head buildup in both aquifers, calculated using our analytical solution in comparison with the results of *Maas* [1987b] and *Hemker and Maas* [1987] (Table 1 of *Maas* [1987b]). (Note that *Maas* [1987b] and *Hemker and Maas* [1987] used $Q = -400\ \pi$ m^3 d^{-1} for pumping-induced drawdown analysis, which we converted into the equivalent case of head buildup from injection.) The agreement between the solutions is excellent. The head buildup in the injection aquifer increases rapidly after the onset of injection and then reaches a steady state value after ~1 d. The start of head buildup in the upper aquifer is slightly delayed compared to the injection aquifer, because the head buildup due to injection needs to propagate through the lower-permeability aquitard. However, a steady state behavior is also reached after ~1 d. The fact that the system approaches steady state so quickly can mainly be attributed to the high aquitard hydraulic conductivity (or small aquitard thickness) corresponding to the parameter choice $B'/K' = 100$ d, as well as the imposed boundary condition of zero head change at the top and bottom of the domain.

4.3. Solution Verification for Coupled Diffuse and Focused Leakage

[70] A coupled diffuse and focused leakage problem is achieved by introducing a leaky well into the two-aquifer–three-aquitard system presented in the previous section. The leaky well is located 25 m away from the injection well along the same line as the injection and observation wells. The leaky well has a radius of 0.15 m and a hydraulic conductivity 1000 times higher than the hydraulic conductivity of the aquifers. Since no existing analytical solutions are available for solution verification of this coupled diffuse and focused leakage example, we employed numerical simulation results for single-phase flow obtained using the COMSOL multiphysics software. A three-dimensional simulation domain was generated with lateral dimensions of 6000 m × 6000 m. This lateral model extent proved to be sufficient, since no boundary effects were observable within the simulation time period.

[71] Figure 5 shows the head buildup in the injection aquifer and the overlying aquifer at the observation well, located between the injection well (10 m away) and the leaky well (15 m away). The agreement between the two solutions is very reasonable, indicating that the generalized analytical solution for coupled diffuse and focused leakage works well. Compared to the case with only diffuse leakage, introduction of a leaky well gives rise to a slightly reduced head buildup in the bottom aquifer together with an earlier and slightly stronger head buildup in the upper aquifer. It appears, however, that the effect of well leakage is not very significant in the example. This is because of the high aquitard conductivity selected in the example case by *Maas* [1987b] and *Hemker and Maas* [1987], which makes diffuse leakage more dominant than it would typically be. More realistic hydrogeologic and geometric properties of a multilayered system are presented in the following section.

5. Application Example

[72] To demonstrate the capability of our generalized analytical solution, we applied it to a more complex example

involving a confined multilayered system consisting of eight aquifers ($N = 8$) and seven alternating aquitards, with one injection well ($N_I = 1$) and one leaky well ($N_L = 1$). The solution domain extends infinitely in the horizontal direction and has no-flow boundary conditions at the top and the bottom. Each of the aquifers is homogeneous, isotropic, and 60 m thick, with a hydraulic conductivity of 2×10^{-1} m d^{-1} (i.e., a permeability of 10^{-13} m^2) and a specific storativity of 1.89×10^{-6} m^{-1}. Each of the seven aquitards, also assumed homogeneous and isotropic, has a thickness of 100 m, a hydraulic conductivity of 2×10^{-6} m d^{-1} (i.e., a permeability of 10^{-18} m^2), and a specific storativity of 7.37×10^{-7} m^{-1} [*Zhou et al.*, 2009]. The hydrogeologic properties for aquifers and aquitards are representative of deep sedimentary basins with alternating sandstone and shale formations, some of which have recently been investigated for geological storage of CO_2 [*National Energy Technology Laboratory*, (*NETL*), 2010].

[73] To demonstrate the impact of pressure perturbation and its effect on coupled diffuse and focused leakage, we assumed that large-scale fluid injection takes place in the center of the domain. Injection occurs into the bottom aquifer at a rate of 5700 m^3 d^{-1} (i.e., 2.08×10^6 m^3 yr^{-1}) for 30 yr. (These injection parameters, as well as the formation properties, are similar to a numerical simulation study by *Birkholzer et al.* [2009], in which the pressure effects generated from industrial-scale injection of CO_2 were investigated.) The leaky well, open to all eight aquifers, has a radius of 0.15 m and is located 2000 m away from the injection well. For sensitivity analysis, we used four different values of hydraulic conductivity for the leaky well (2×10^5 m d^{-1}, referred to as the base case, as well as 2×10^4 m d^{-1}, 2×10^3 m d^{-1}, and zero). An observation well is located at a distance of 1990 m from the injection well and 10 m from the leaky well, with all three wells arranged along a horizontal line. For simplification, we assume that the density and viscosity of groundwater (thus, hydraulic conductivity and specific storativity) do not change in space and time, even though the entire domain is 1180 m from top to bottom. We also assume that the injected fluid has the same properties as the native groundwater.

[74] Figure 6 shows results from the new analytical solutions in the form of vertical profiles of head buildup at 30 (end of injection) and 100 yr (70 yr after the end of injection). As can be observed from the zero well conductivity case, diffuse leakage alone leads to head buildup in aquifers 2, 3, and 4 and aquitards 1, 2, and 3 at the end of the fluid injection (Figure 6a). Notice the quasi-linear head profile in each of the perturbed aquitards, which is in contrast with early-time behavior where steep gradients can be observed at aquifer–aquitard interfaces (not shown in the figure). This is because the time for head propagation through aquitards, which can be calculated using equation (10), is only about 1.6 yr. The head-buildup profiles change significantly with increasing hydraulic conductivity of the leaky well, showing relatively less perturbation in the injection formation

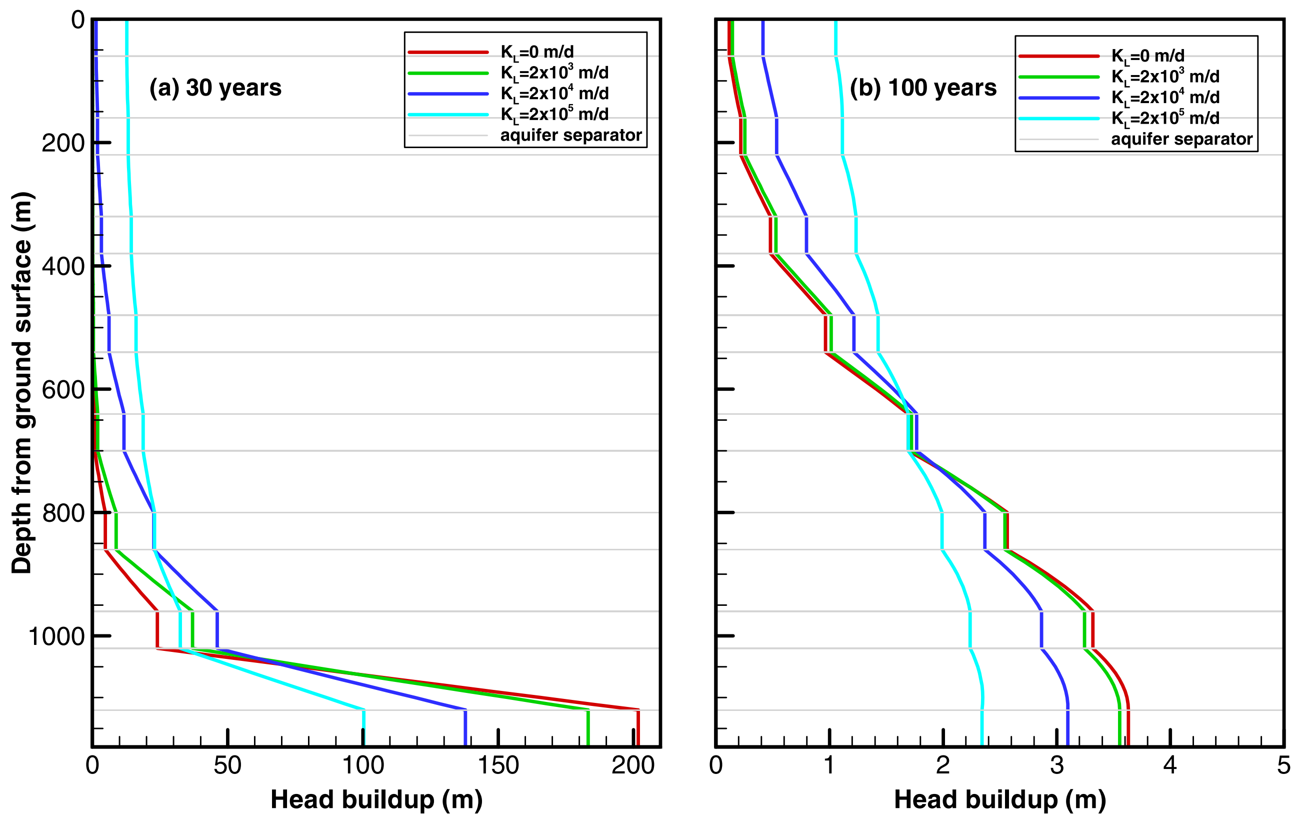

Figure 6. Vertical profiles of hydraulic head buildup (m) at the observation well as a function of hydraulic conductivity of the leaky well at (a) 30 and (b) 100 yr.

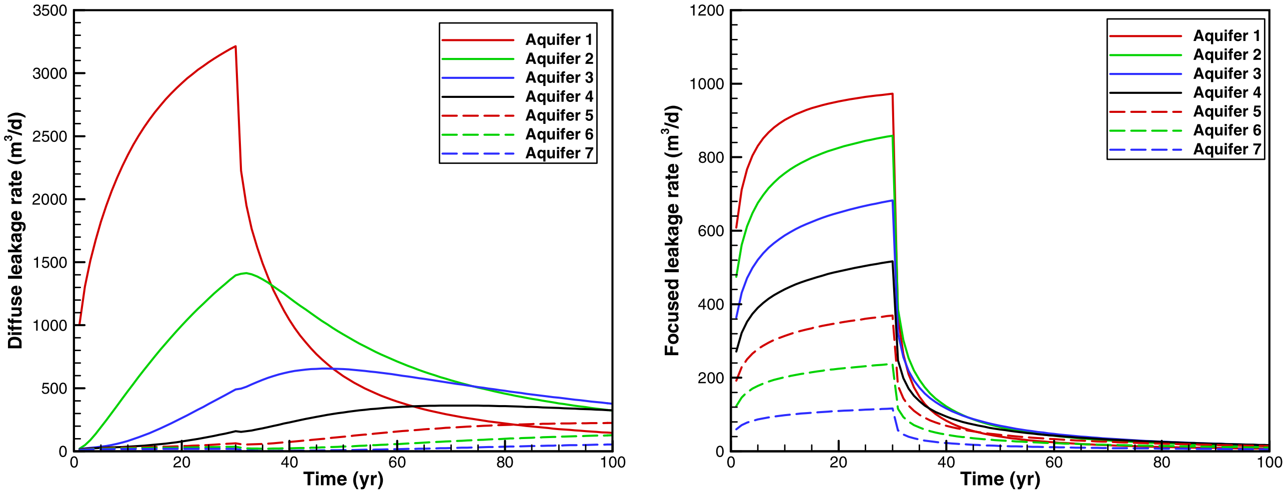

Figure 7. Comparison of diffuse and focused leakage rate ($m^3\ d^{-1}$) of groundwater through the top of aquifers 1–7, as a function of time. Aquifer 1 is the bottom aquifer in which injection occurs. The leaky well hydraulic conductivity is 2×10^5 m d^{-1}.

(aquifer 1) combined with more extensive and stronger buildup in overlying aquifers. For the highest well conductivity (2×10^5 m d^{-1}), the maximum buildup in the injection aquifer decreases by ~50% compared to the case without a leaky well; and head changes propagate all the way up to the top aquifer, indicating that in this case deep, saline groundwater might be pushed into the shallow aquifer.

[75] As injection stops after 30 yr, the system starts to re-equilibrate, both horizontally (within the aquifers as head changes propagate away from the injection location) and vertically (through diffuse leakage and well leakage into overlying aquifers and aquitards). This process has been almost (but not entirely) completed at 100 yr (Figure 6b), with the high-permeability leaky well case furthest along. This is evident from the head buildup in the bottom aquifer which has now decreased to a small fraction of the maximum buildup observed during injection, and from the small head changes that can now be seen over the entire vertical extent of the domain.

[76] Figure 7 contrasts the flow rates from diffuse and focused leakage at aquifer–aquitard interfaces, here showing the base case with a hydraulic conductivity of 2×10^5 m d^{-1} for the leaky well. Despite the high well hydraulic conductivity, the diffuse leakage rate out of the injection formation (aquifer 1) exceeds the focused leakage rate by a factor of approximately three. At the end of injection, diffuse and focused leakage rates together account for ~75% of the injection rate, indicating how important diffuse pressure bleed-off and leaky wells can be in reducing injection-related head buildup. As can be expected from the permeability contrast and the differences in storativity, the response to system perturbation is much faster for focused leakage compared to diffuse leakage, which can be observed at the beginning of injection and also when injection stops.

[77] The next example results demonstrate the use of the new analytical solutions in a sensitivity evaluation. Figure 8 shows the 30-yr cumulative volume of well leakage leaving the injection aquifer (normalized by the cumulative injected groundwater volume) as a function well conductivity (K_L) for five different cases of aquitard conductivity (K'). (The aquifer conductivity remains unchanged at 0.2 m d^{-1}.) As expected, the cumulative well-leakage volume is strongly affected by well conductivity. While this is true over a wide parameter range, all leakage curves start approaching an asymptotic maximum value at about 10^6 to 10^7 m d^{-1}; further increases in well conductivity do not result in further increases in leakage volume. The cumulative well leakage is also quite sensitive to aquitard conductivity: As aquitards become more permeable, the cumulative flow through the well reduces significantly. This means that the rate of focused leakage through leaky wells and the cumulative leakage volume can be severely overestimated if aquitards with nonzero conductivity would be assumed impermeable [*Javandel et al.*, 1988; *Avci*, 1994; *Nordbotten et al.*, 2004].

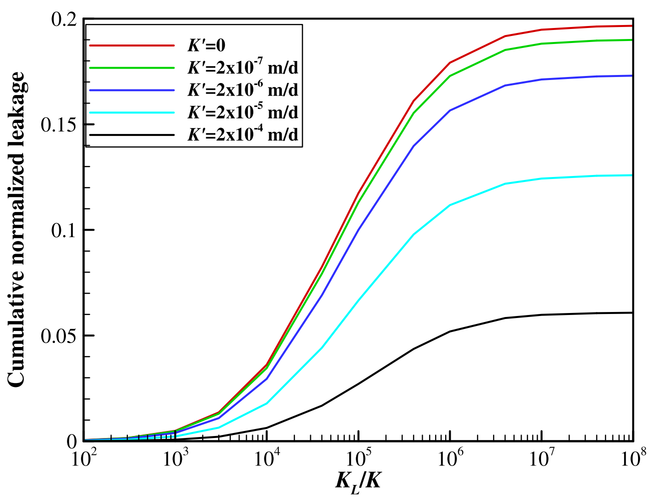

Figure 8. Effect of aquitard conductivity (K') on cumulative leakage through the leaky well from the injection aquifer after 30 yr, with the aquifer conductivity fixed at 0.2 m d^{-1}.

6. Summary and Conclusions

[78] A new set of generalized analytical solutions is presented for coupled diffuse and focused leakage in a multilayered system consisting of any number of aquifers, alternating aquitards, pumping/injection wells, and leaky wells. The paper discusses in detail the governing equations for horizontal flow in the aquifers, which are coupled with each other by the expressions for vertical flow in alternating aquitards and leaky wells, and reviews the initial and boundary conditions, as well as the conditions at aquifer–aquitard interfaces, well–aquifer interfaces, and well–aquitard interfaces. The solution methodology is as follows: First, the governing equations for N aquifers, alternating aquitards, and N_L leaky wells are transformed into the Laplace domain. Second, the diffuse-leakage coupling between the N aquifers, which results in a system of N coupled ordinary differential equations (ODEs) written in a matrix-vector form, is decoupled via eigenvalue analysis. This allows solutions for pressure perturbation in response to injection/pumping at an active well to be obtained independently for each aquifer. Third, employing the superposition method and expressing the boundary conditions at leaky wells through continuity equations results in a system of $N_L \times N$ linear algebraic equations. This equation system is solved to provide expressions for the drawdown due to leaky wells and the unknown leakage rates at the $N_L \times N$ well–aquifer segments. The total head buildup and the rates of focused and diffuse leakage in response to N_I injection wells and N_L leaky wells are then obtained using superposition. For ease of application, a FORTRAN-based software package was developed which reads in the problem specifications, executes the calculation, and returns calculation results. The latter involves numerical inversion to convert results given in the Laplace domain into the corresponding variables in the real time domain.

[79] The new solutions described here encompass and advance the features and capabilities of existing analytical solutions (which are available for diffuse leakage, or for focused leakage, but not for both combined) and provide an important improvement in solving analytically the subsurface flow processes in multilayered aquifer–aquitard systems. Accuracy of the new solutions was verified against existing analytical solutions for diffuse leakage and for focused leakage. Additional verification for coupled diffuse and focused leakage was conducted via comparison against numerical solutions. An example application involving an eight-aquifer system with leaky aquitards and one leaky well illustrated how the new solutions can be useful in evaluating the large-scale perturbations of hydraulic head and fluid flow in response to major fluid injection/pumping operations. The examples also showed the high computational efficiency of the new solutions, even for cases with a large number of alternating aquifers and aquitards, and multiple active and leaky wells.

Appendix A: Solution Procedure

[80] For ease of application, we developed a FORTRAN-based software package that reads in the problem specifications, executes the analytical solution, and calculates (1) the time-dependent hydraulic head buildup in aquifers and aquitards, (2) the transient rate of diffuse leakage through aquifer–aquitard interfaces, and (3) the transient rate of focused leakage through leaky wells. The software package includes the following solution steps:

[81] 1. Assemble the $N \times N$ diffuse-leakage-coupling matrix, **A**, for the coupled system of ODEs by calculating its tri-diagonal coefficients based on equations (13b)–(13d), (11c), and (10b), using specified hydrogeologic and geometric properties of aquifers and aquitards, and the conditions at the top and bottom boundaries.

[82] 2. Decompose matrix **A** into eigenvalue and eigenvectors using equation (14) and a linear eigensystem solver subroutine in the Fortran IMSL library. Because the diffuse-leakage terms $f_i^\alpha(p)$, $g_i^\alpha(p)$ are functions of transform variable p, the development and decomposition of matrix **A** are needed only once at a given time, no matter how many injection wells and leaky wells are involved. The eigenvalue system analysis is not required when all alternating aquitards are impervious; in this case, **A** is a diagonal matrix, and no coupling between different aquifers is caused by diffuse leakage.

[83] 3. Superpose the head buildup induced by injection, with a known rate, at each injection well and by leakage, with an unknown rate, at each well–aquifer segment of all leaky wells to calculate the total head buildup under all influences using equation (36).

[84] 4. Develop the system of $N \times N_L$ linear algebraic equations for all well–aquifer segments involving all leaky wells using the superposed total head buildup, on the basis of equations (30) and (31); solve the resulting equations for the unknown coefficients $c_{j,l}^L$ using linear system solvers in the Fortran IMSL library. With diffuse leakage, the focused-leakage coupling matrix, **B**, is a full matrix.

[85] 5. Calculate the diffuse leakage rates through aquifer–aquitard interfaces using equations (11b) and (11c), and interface-integrated leakage rates using equation (39) under influences of all active and leaky wells; calculate the head buildup in aquitards using equation (10), with the solved head buildup in neighboring aquifers, when diffuse leakage is involved.

[86] 6. Numerically invert these solutions in the Laplace domain to obtain their corresponding variables (i.e., head buildup in aquifers and aquitards, diffuse leakage rates, and focused leakage rates, as well as their cumulative rates) in the real time domain using the Stehfest numerical Laplace inversion method [*Stehfest*, 1970a, 1970b]. This method is well suited for groundwater hydraulics problems [*Hemker and Maas*, 1987; *Cheng and Morohunfola*, 1993]. In the Stehfest method, the solution in a real time domain is approximated by a finite series. There is a tradeoff between the number of terms in the series and the round-off error. *Hemker and Maas* [1987] suggested using 10 terms in the series, while we found that using 16 terms produced accurate and efficient calculations in our applications. For strong changes in injection/pumping rates and multiple-injection/pumping periods, we employ a superposition method in time by adding separate smooth solutions for each period interval with constant-rate injection/pumping. In this way, possible errors resulting from using the Stehfest method for cases with abrupt changes are avoided.

[87] Execution of all above steps is only needed for a problem involving both diffuse and focused leakage. Step 4 is not required for a system without leaky wells. Steps 1, 2, and 5 can be omitted for a system with impervious

aquitards. The solution procedure is very efficient, and computational time for each of the verification and application examples is <1 s. The FORTRAN code developed in this paper can be obtained from the authors upon request.

[88] **Acknowledgments.** The authors wish to thank Hailong Li and one anonymous reviewer, as well as James E. Houseworth of Lawrence Berkeley National Laboratory (LBNL), for their careful review of the manuscript and the suggestion of improvements. This work was funded in part by the USEPA, Office of Water and Office of Air and Radiation, under an Interagency Agreement with the U. S. Department of Energy (USDOE) at LBNL. Supplementary funding was provided by the Assistant Secretary for Fossil Energy, Office of Sequestration, Hydrogen, and Clean Coal Fuels, through the National Energy Technology Laboratory, under the USDOE contract DE-AC02-05CH11231.

References

Avci, C. (1994), Evaluation of flow leakage through abandoned wells and boreholes, *Water Resour. Res.*, *30*(9), 2565–2578, doi:10.1029/94WR00952.

Bear, J. (1972), *Dynamics of Fluids in Porous Media*, 784 pp., Dover, New York.

Bear, J. (1979), *Hydraulics of Groundwater*, 592 pp., Dover, New York.

Birkholzer, J. T., and Q. Zhou (2009), Basin-scale hydrogeologic impacts of CO_2 storage: Regulatory and capacity implications, *Int. J. Greenhouse Gas Control*, *3*(6), 745–756.

Birkholzer, J. T., Q. Zhou, C. F. Tsang (2009), Large-scale impact of CO_2 storage in deep saline aquifers: A sensitivity study on the pressure response in stratified systems, *Int. J. Greenhouse Gas Control*, *3*(2), 181–194.

Bredehoeft, J. D., C. E. Neuzil, and P. C. D. Milly (1983), Regional flow in the Dakota aquifer: A study of the role of confining layers, *U.S. Geol. Surv. Water Supply Pap.*, pp. 45:2237.

Celia, M. A., S. Bachu, J. M. Nordbotten, S. Gasda, and H. Dahle (2004), Quantitative estimation of CO_2 leakage from geological storage: Analytical models, numerical models, and data needs, in *7th International Conference on Greenhouse Gas Control Technologies* (*GHGT-7*), edited by E. S. Rubin, D. W. Keith, and C. F. Gilboy, Vancouver, Canada.

Celia, M. A., J. M. Nordbotten, B. Court, M. Dobossy, and S. Bachud (2011), Field-scale application of a semi-analytical model for estimation of CO_2 and brine leakage along old wells, *Int. J. Greenhouse Gas Control*, *5*(2), 257–269.

Cheng, A. H.-D. (1994), Reply to "Comment on 'Multilayered leaky aquifer systems: 1. Pumping well solutions' by A. H.-D. Cheng and O. K. Morohunfola", *Water Resour. Res.*, *30*(11), 3231, doi:10.1029/94WR01803.

Cheng, A. H.-D., and O. K. Morohunfola (1993), Multilayered leaky aquifer systems: Pumping well solutions, *Water Resour. Res.*, *29*(8), 2787–2800, doi:10.1029/93WR00768.

Gass, T. E., J. H. Lehr, and H. W. Heiss Jr. (1977), Impact of abandoned wells on ground water, *Rep. EPA-600/3-77-095*, pp. 53, US Environ. Prot. Agency, Washington, D. C.

Hantush, M. S. (1960), Modification of the theory of leaky aquifers, *J. Geophys. Res.*, *65*(11), 3713–3725, doi:10.1029/JZ065i011p03713.

Hantush, M. S., and C. E. Jacob (1955), Nonsteady radial flow in an infinite leaky aquifer, *Eos Trans. AGU*, *36*(1), 95–100.

Hart, D. J., K. R. Bradbury, and D. T. Feinstein (2006), The vertical hydraulic conductivity of an aquitard at two spatial scales, *Ground Water*, *44*(2), 201–211.

Hemker, C. J. (1985), Transient well flow in leaky multiple-aquifer systems, *J. Hydrol.*, *81*, 111–126.

Hemker, C. J., and C. Maas (1987), Unsteady flow to wells in a layered and fissured aquifer systems, *J. Hydrol.*, *90*, 231–249.

Hemker, C. J., and C. Maas (1994), Comment on "Multilayered leaky aquifer systems: 1. Pumping well solutions" by A. H.-D. Cheng, and O. K. Morohunfola, *Water Resour. Res.*, *30*(11), 3229–3230, doi:10.1029/94WR01802.

Hemker, C. J., and V. E. A. Post (2011), MLU for Windows—Well flow modeling in multilayer aquifer systems, 30 pp., Amsterdam, Netherlands, available at http://www.microfem.com.

Huisman, L., and J. Kemperman (1951), Bemaling van Spanningsgrondwater, *De Ingenieur*, *62*, B29–B35 (in Dutch).

Hunt, B. (1985), Flow to a well in a multiaquifer system, *Water Resour. Res.*, *21*, 1637–1641, doi:10.1029/WR021i011p01637.

Javandel, I., C. F. Tsang, P. A. Whiterspoon, and D. Morganwalp (1988), Hydrologic detection of abandoned wells near proposed injection wells for hazardous waste disposal, *Water Resour. Res.*, *24*(2), 261–270, doi:10.1029/WR024i002p00261.

Konikow, L. F., and C. E. Neuzil (2007), A method to estimate groundwater depletion from confining layers, *Water Resour. Res.*, *43*, W07417, doi:10.1029/2006WR005597.

Lesage, S., R. E. Jackson, M. Priddle, P. Beck, and K. G. Raven (1991), Investigation of possible contamination of shallow groundwater by deeply injected liquid industrial wastes, *Ground Water Monit. Rev.*, *11* (winter), 151–159.

Maas, C. (1986), The use of matrix differential calculus in problems of multiple-aquifer flow, *J. Hydrol.*, *88*, 43–67.

Maas, C. (1987a), Groundwater flow to a well in a layered porous medium, 1. Steady flow, *Water Resour. Res.*, *23*(8), 1675–1681, doi:10.1029/WR023i008p01675.

Maas, C. (1987b), Groundwater flow to a well in a layered porous medium, 2. Nonsteady multiple-aquifer flow, *Water Resour. Res.*, *23*(8), 1683–1688, doi:10.1029/WR023i008p01683.

Moench, A. F. (1985), Transient flow to a large-diameter well in an aquifer with storative semiconfining layers, *Water Resour. Res.*, *21*(8), 1121–1131, doi:10.1029/WR021i008p01121.

National Energy Technology Laboratory (NETL) (2010), *Carbon Sequestration Atlas of the United States and Canada, Third Edition*, National Energy Technology Laboratory Report, December 2010, available at http://www.netl.doe.gov/technologies/carbon_seq/refshelf/atlas/.

Neuman, S. P., and P. A. Witherspoon (1969), Theory of flow in a two-aquifer system, *Water Resour. Res.*, *5*(4), 803–816, doi:10.1029/WR005i004p00803.

Neuzil, C. E. (1986), Groundwater flow in low-permeability environments, *Water Resour. Res.*, *22*(8), 1163–1195, doi:10.1029/WR022i008p01163.

Neuzil, C. E. (1994), How permeable are clays and shales?, *Water Resour. Res.*, *30*(2), 145–150, doi:10.1029/93WR02930.

Nicot, J. P. (2008), Evaluation of large-scale carbon storage on fresh-water section of aquifers: A Texas study, *Int. J. Greenhouse Gas Control*, *2*(4), 582–593.

Nicot, J. P. (2009), A survey of oil and gas wells in the Texas Gulf Coast, USA, and implications for geological sequestration of CO_2, *Environ. Geol.*, *57*, 1625–1638, doi:10.1007/s00254-008-1444-4.

Nordbotten, J. M., M. A. Celia, and S. Bachu (2004), Analytical solutions for leakage rates through abandoned wells, *Water Resour. Res.*, *40*, W04204, doi:10.1029/2003WR002997.

Stehfest, H. (1970a), Numerical inversion of Laplace transforms, *Commun. ACM*, *13*(1), 47–49.

Stehfest, H. (1970b), Remark on algorithm. Numerical inversion of Laplace transforms, *Commun ACM*, *13*(10), 624.

Theis, C. V. (1935), The relation between the lowering of the piezometric surface head and the rate and duration of discharge of a well using ground-water storage, *Eos Trans. AGU*, *16*, 519–524.

Veling, E. J. M., and C. Maas (2009), Strategy for solving semi-analytically three-dimensional transient flow in a coupled *N*-layered aquifer system, *J. Eng. Math.*, *64*, 145–161, doi:10.1007/s/10665-008-9256-9.

Yang, Y., and A. C. Aplin (2007), Permeability and petrophysical properties of 30 natural mudstones, *J. Geophys. Res.*, *112*, B03206, doi:10.1029/2005JB004243.

Yang, Y., and A. C. Aplin (2010), A permeability–porosity relationship for mudstones, *Mar. Pet. Geol.*, *27*, 1692–1697.

Young, H. L. (1992), Summary of ground-water hydrology of the Cambrian-Ordovician aquifer system in the northern Midwest, United States, *U. S. Geol. Surv. Prof. Pap.*, 1405-A.

Zhou, Q., and J. T. Birkholzer (2011), On scale and magnitude of pressure build-up induced by large-scale geologic storage of CO_2, *Greenhouse Gases: Science and Technology*, *1*, 11–20, doi:10.1002/ghg3.001.

Zhou, Q., J. T. Birkholzer, and C. F. Tsang (2009), A semi-analytical solution for large-scale injection-induced pressure perturbation and leakage in a laterally bounded aquifer-aquitard system, *Transp. Porous Media*, *78*(1), 127–148.

Zhou, Q., J. T. Birkholzer, C. F. Tsang, and J. Rutqvist (2008), A method for quick assessment of CO_2 storage capacity in closed and semi-closed saline aquifers, *Int. J. Greenhouse Gas Control*, *2*, 626–639.

Zhou, Q., J. T. Birkholzer, E. Mehnert, Y. F. Lin, and K. Zhang (2010), Modeling basin- and plume-scale processes of CO_2 storage for full-scale deployment, *Ground Water*, *48*(4), 494–514.

J. T. Birkholzer, A. Cihan, and Q. Zhou, Earth Sciences Division, Lawrence Berkeley National Laboratory, 1 Cyclotron Rd., Mail Stop 90-1116, Berkeley, CA 94720, USA. (acihan@lbl.gov)