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Spherical Wave Propagation in a Nonlinear Elastic Medium

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Abstract

Nonlinear propagation of spherical waves generated by a point-pressure source is considered for the cases of monochromatic and impulse primary waveforms. The nonlinear five-constant elastic theory advanced by Murnaghan is used where general equations of motion are put in the form of vector operators, which are independent of the coordinate system choice. The ratio of the nonlinear field component to the primary wave in the far field is proportional to $ln(r)$ where r is a propagation distance. Near-field components of the primary field do not contribute to the far field of nonlinear component.

1 Introduction

This work was done in 1993 during a multi-lab DOE project on nonlinear seismic effects. The author's contribution back then resulted in two reports with theoretical results (Korneev et al., 1998; Korneev, 1998). The experimental LBNL results are covered in Daley et al. (1992). By the end of the project, the results for a point-pressure source had neither a quantitative part nor data and were abandoned until now. Currently, there is a renewed interest to nonlinear phenomenon in seismic prospecting. Some publications demonstrate correlations between the observed nonlinear properties of seismic fields and hydrocarbon reservoir characteristics (Zhukov et al.,2008). In this connection, the results of the report might be of interest for geophysicists dealing with seismic nonlinear phenomenon. Another possible application of the considered problem is the physics of powerful underground explosions. The results of this report take into account the elastic nonlinearity within a framework of the five-constant Murnaghan

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theory (Murnaghan, 1951), and the relative smallness of the nonlinear field component compared to the linear one. The author assumes that the basic equations and expressions of this theory can be found in several studies (Zarembo and Krasil'nikov, 1971,1966; Taylor and Rollins, 1999; Landau and Lifshitz, 1953;Hughes and Kelly, 1953; Gedroits and Krasil'nikov, 1963) and the other sited publications.

2 Equations of motion

Following the work of Jones and Kobett (1963), we will use the equation of motion in an perfectly elastic solid

$$\rho \frac{\partial^2 U_i}{\partial t^2} \; - \; \mu \frac{\partial^2 u_i}{\partial x_k^2} \; - \; (\lambda \, + \, \mu) \frac{\partial^2 u_k}{\partial x_k \partial x_i} \; = \; F_i \tag{1}$$

where F_i, the i^{ith} component of $\mathbf{F}$, has a value of the second order in smallness and is given by:

$$F_i \; = \; C_1 \left(\frac{\partial^2 u_s}{\partial x_k^2} \frac{\partial u_s}{\partial x_i} \; + \; \frac{\partial^2 u_s}{\partial x_k^2} \frac{\partial u_i}{\partial x_s} \; + \; 2 \frac{\partial^2 u_i}{\partial x_s \partial x_k} \frac{\partial u_s}{\partial x_k} \right) + C_2 \left(\frac{\partial^2 u_s}{\partial x_i \partial x_k} \frac{\partial u_s}{\partial x_k} \; + \; \frac{\partial^2 u_k}{\partial x_s \partial x_k} \frac{\partial u_i}{\partial x_s} \right)$$

$$+ \, C_3 \frac{\partial^2 u_i}{\partial x_k^2} \frac{\partial u_s}{\partial x_s} + C_4 \left(\frac{\partial^2 u_k}{\partial x_s \partial x_k} \frac{\partial u_s}{\partial x_i} \; + \; \frac{\partial^2 u_s}{\partial x_i \partial x_k} \frac{\partial u_k}{\partial x_s} \right) \; + \; C_5 \frac{\partial^2 u_k}{\partial x_i \partial x_k} \frac{\partial u_s}{\partial x_s} \tag{2}$$

where constants

$$C_1 \; = \; \mu + \frac{A}{4} \; , \qquad C_2 \; = \; \lambda + \mu + \frac{A}{4} + B \; , \qquad C_3 \; = \; \lambda + B \tag{3}$$

$$C_4 \; = \frac{A}{4} + B \; , \quad C_5 \; = \; B \, + \, 2C \tag{4}$$

contain the nonlinear constants A, B, C by Landau and Lifshitz (1953), which can be expressed through the constants introduced by Murnaghan (1951).

After some algebra, the nonlinear term (2) can be represented in the general form:

$$\mathbf{F} \; = \; C_1 \mathbf{W}_1 \; + \; C_2 \mathbf{W}_2 \; + \; C_3 \mathbf{W}_3 \; + \; C_4 \mathbf{W}_4 \; + \; C_5 \mathbf{W}_5 \tag{5}$$

where

$$\mathbf{W}_1 \; = \; [\Delta \mathbf{u} \times rot \mathbf{u}] \; + \; \frac{1}{2} \nabla \Delta(\mathbf{uu}) \; + \; div \mathbf{u} \, \Delta \mathbf{u} \; - \; \Delta [u \times rot \mathbf{u}] \; + \; rot[\mathbf{u} \times \Delta u] \; - \; \mathbf{u} \, \Delta div \mathbf{u} \tag{6}$$

$$\mathbf{W}_2 \ = \ \frac{1}{2}(\mathbf{W} \ + \ \nabla(rotrot\mathbf{u}\cdot\mathbf{u}) \ - \ rot[\nabla div\mathbf{u}\times\mathbf{u}] \ - \ [\nabla div\mathbf{u}\times rot\mathbf{u}]) \tag{7}$$

$$\mathbf{W}_3 \ = \ div\mathbf{u}\,\Delta\mathbf{u} \tag{8}$$

$$\mathbf{W}_4 \ = \ \frac{1}{2}(\mathbf{W} \ + \ [\nabla div\mathbf{u}\times rot\mathbf{u}] \ - \ rot[\nabla div\mathbf{u}\times\mathbf{u}] \ - \ \nabla div[\mathbf{u} times rot\mathbf{u}]) \tag{9}$$

$$\mathbf{W}_5 \ = \ div\mathbf{u}\,\nabla div\mathbf{u} \tag{10}$$

$$\mathbf{W} \ = \ \frac{1}{2}\nabla\Delta(\mathbf{uu}) \ - \ \mathbf{u}\,\Delta div\mathbf{u} \ + \ div\mathbf{u}\,\nabla div\mathbf{u} \tag{11}$$

independent of a choice of coordinate system.

3 Spherical Symmetry

In the case of a full spherical symmetry, we have radial displacement only, and the equation of motion (1) can be reduced to a scalar form

$$\frac{d^2u}{dr^2} \ + \ \frac{2}{r}\frac{du}{dr} \ - \ \frac{2}{r^2}u \ - \ \frac{1}{v^2}\frac{d^2u}{dt^2} \ = \ -\frac{F}{\lambda+2\mu}, \qquad v \ = \ \sqrt{\frac{\lambda+2\mu}{\rho}} \tag{12}$$

$$\begin{aligned} F \ &= \ D\left(\frac{du}{dr} \ + \ \frac{2}{r}u\right)\left(\frac{d^2u}{dr^2} \ + \ \frac{2}{r}\frac{du}{dr} \ - \ \frac{2}{r^2}u\right) \\ &+ \ E\left(\frac{d^2u}{dr^2}\frac{du}{dr} \ + \ \frac{1}{r}\left(\frac{du}{dr}\right)^2 \ - \ \frac{u^2}{r^3}\right) \end{aligned} \tag{13}$$

where we use the notations

$$D \ = \ \lambda + 2B \ + \ 2C \ = \ \lambda \ + \ 2l \tag{14}$$

$$E \ = \ 2(\lambda \ + \ 3\mu \ + \ A \ + \ 2B) \ = \ 2(\lambda \ + \ 3\mu \ + \ 2m) \tag{15}$$

with nonlinear constants l and m by Murnaghan (1951). The similar equation was obtained in Beresnev (1990) by direct transformation of equation (2) where the numerical finite difference scheme was used to solve it. Here we develop an analytical approach using the relative smallness of F, which allows one to seek the solution in the form

$$u \;=\; u_0 \;+\; u_1 \tag{16}$$

where u_0 is the solution of reduced equation (12) when F is assumed equal to zero. Putting (16) into (12) and assuming that $|u_0| \;>>\; |u_1|$ we obtain a linear equation for u_1, where the left-hand side contains components of u_1 and the right-hand side F depends exclusively on the previously determined function u_0. We also can take an advantage of u_0 being the solution of the reduced equation (12) to simplify the equation (13) for F .

$$F \;=\; D\left(\frac{du_0}{dr} \;+\; \frac{2}{r}u_0\right)\frac{1}{v^2}\frac{d^2u_0}{dt^2} \;+\; E\left(\frac{d^2u_0}{dr^2}\frac{du_0}{dr} \;+\; \frac{1}{r}\left(\frac{du_0}{dr}\right)^2 \;-\; \frac{u_0^2}{r^3}\right) \tag{17}$$

Two time-dependence cases for the primary field u_0 will be considered: monochromatic and the impulse like. In the next section, this method will be applied to the problem of propagating monochromatic elastic spherical waves in an isotropic homogeneous nonlinear medium.

4 Monochromatic Primary Wave

We seek the solution for the primary wave

$$u_0 \;=\; I\frac{sin(\omega t \;-\; z) \;+\; zcos(\omega t \;-\; z)}{rz} \;, \quad z \;=\; \frac{\omega r}{v} \tag{18}$$

where I is an arbitrary constant. After substitution of (18) in (17), function F has the form

$$F \;=\; F_0(r) \;+\; F_s(r)\,sin2(\omega t \;-\; z) \;+\; F_c(r)\,cos2(\omega t \;-\; z) \tag{19}$$

where functions $F_0(r)$, $F_s(r)$, $F_c(r)$ are finite sums of negative powers of r

$$F_0(r) \;=\; -I^2D\frac{k^2}{2r^3} \;-\; I^2E\frac{k^3}{2r^2}\left(\frac{1}{z^3} \;+\; \frac{9}{z^5}\right) \tag{20}$$

$$F_c(r) \;=\; I^2\frac{k^2}{r^3}\left[\frac{D}{2} \;+\; E\left(2 - \frac{17}{2z^2} \;+\; \frac{9}{2z^4}\right)\right] \tag{21}$$

$$F_s(r) \;=\; I^2\frac{k^3}{r^2}\left[-\frac{D}{2} \;+\; E\left(-\frac{1}{2} + \frac{5}{z^2} - \frac{9}{z^4}\right)\right] \tag{22}$$

The structure of (19) allows us to seek the solution for u_1 in the form

$$u_1(r,t) \;=\; u_1^0(r) \;+\; Re\left(e^{2i\omega t}\tilde{u}_1(r)\right) \tag{23}$$

where function $u_1^0(r)$ satisfies the equation

$$\frac{d^2u_1^0}{dr^2} \;+\; \frac{2}{r}\frac{du_1^0}{dr} \;-\; \frac{2}{r^2}u_1^0 \;=\; -\frac{F_0}{\lambda+2\mu} \tag{24}$$

and for the complex function $\tilde{u}_1(r)$, we have inhomogeneous spherical Bessel equation:

$$\frac{d^2\tilde{u}_1}{dr^2} \;+\; \frac{2}{r}\frac{d\tilde{u}_1}{dr} \;+\; \left(4k^2 \;-\; \frac{2}{r^2}\right)\tilde{u}_1 \;=\; -\frac{F_c \;-\; iF_s}{\lambda+2\mu}\,e^{-2iz}, \quad k \;=\; \frac{\omega}{v} \tag{25}$$

Solutions for equation (25) when its right-hand side has the form $\frac{e^{-ikr}}{r^n}$, $n = 2,3,4,5,6,7$ can be found in Appendix A., which after some algebra enable us to obtain the expression for the nonlinear field:

$$u_1 \;=\; -\frac{I^2Dk^3}{\lambda\,+\,2\mu}\left(\frac{1}{4z} \;+\; u_1^{(1)}(z)\right) \;-\; \frac{I^2Ek^3}{\lambda\,+\,2\mu}\left(-\frac{1}{8z^3} \;-\; \frac{1}{4z^5} \;+\; u_1^{(2)}(z)\right) \tag{26}$$

where

$$u_1^{(1)}(z) \;=\; \frac{1}{4}Re\left(e^{2i\omega t}\left[h_1^{(1)}(2z)Ei(-i4z) \;+\; h_1^{(2)}(2z)\ln 2z\right]\right) \tag{27}$$

$$u_1^{(2)}(z) \;=\; \frac{1}{4}Re\left(e^{2i\omega t}\left[h_1^{(1)}(2z)Ei(-i4z) \;+\; h_1^{(2)}(2z)\ln 2z \;+\; \left(-\frac{3i}{2} \;-\; \frac{5}{2z} \;+\; \frac{2i}{z^2} \;+\; \frac{1}{z^3}\right)\frac{e^{-2iz}}{z^2}\right]\right) \tag{28}$$

with first index spherical Hankel functions

$$h_1^{(1)}(z) \;=\; -\frac{i\,+\,z}{z^2}e^{iz}\;, \quad h_1^{(2)}(z) \;=\; \frac{i\,-\,z}{z^2}e^{-iz} \tag{29}$$

of the first and second kind, respectively. Function $Ei(-ix)$ is defined by

$$Ei(-ix) \;=\; -\int_x^{\infty}\frac{e^{-i\xi}}{x}id\xi \tag{30}$$

Any function of the form

$$u_1^{(0)} \;=\; \frac{a_0}{z^2} \;+\; a_1 \frac{sin(\omega t \,-\, z) \;+\; zcos(\omega t \,-\, z)}{rz} \;+\; a_2 \frac{cos(\omega t \,-\, z) \;-\; zsin(\omega t \,-\, z)}{rz} \tag{31}$$

with arbitrary coefficients $a_0, a_1,\, a_2$ may be added to the solution (26) to match boundary conditions at reference radius R . In the far field, when $z \;>>\; 1$, the nonlinear field (26) has the asymptotic value

$$u_1 \;=\; I^2 \frac{\omega^2}{v^2} \frac{(3\lambda \,+\, 6\mu \,+\, 2A \,+\, 6B \,+\, 2C)}{8(\lambda \,+\, 2\mu)} \frac{ln(r/R)}{r} cos2(\omega t \,-\, z) \tag{32}$$

Assuming that $x \,=\, r - R \;<<\; R$, we obtain from (32):

$$u_1 \;=\; I^2 \frac{\omega^2}{v^2} \frac{(3\lambda \,+\, 6\mu \,+\, 2A \,+\, 6B \,+\, 2C)}{8(\lambda \,+\, 2\mu)} \frac{x}{R^2} cos2(\omega t \,-\, z) \tag{33}$$

which differs from the result for the plane compressional primary wave (Polyakova, 1964) by a factor of $\frac{1}{R^2}$.

The obtained analytical solutions of equation (24) are verified by comparison with its finite-difference solutions.

5 Impulse Primary Wave

Here we consider the nonlinear propagation of the pulse generated by the primary wave

$$u_0 \;=\; \frac{q_{,\tau}(\tau)}{vr} \;+\; \frac{q(\tau)}{r^2} \;, \quad \tau \;=\; t \,-\, \frac{r}{v} \tag{34}$$

where source function $q(\tau)$ describes an initial waveform. Substituting (34) in (17) and taking the inverse Fourier transform over t we have

$$f(\omega) \;=\; \frac{1}{\sqrt{2pi}} \int_{-\infty}^{\infty} F(t) e^{-i\omega t} dt \;=\; -\, \frac{D}{\lambda \,+\, 2\mu} \frac{Q_2}{v^4 r^3} e^{-ikr} (1 \,+\, \frac{ikr}{2})$$

$$-\; \frac{E}{\lambda \,+\, 2\mu} \left[\frac{ik}{2v^4 r^2} Q_2 \;+\; \frac{1}{v^2} \left(-\frac{k^2}{r^3} \,+\, \frac{ik}{r^4} \,+\, \frac{1}{r^5} \right) Q_1 \;+\; \left(-\frac{ik^3}{r^4} \,-\, \frac{4k^2}{r^5} \,+\, \frac{9ik}{r^6} \,+\, \frac{9}{r^7} \right) Q_0 \right] e^{-ikr} \tag{35}$$

Functions Q_0, Q_1, and Q_2 from (35) depend on waveform $q(\tau)$ of the primary signal:

$$Q_0 \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q^2(\tau) e^{-i\omega\tau} d\tau \tag{36}$$

$$Q_1 \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q_{,\tau}^2(\tau) e^{-i\omega\tau} d\tau \tag{37}$$

$$Q_2 \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q_{,\tau\tau}^2(\tau) e^{-i\omega\tau} d\tau \tag{38}$$

Therefore, in a frequency domain, a nonlinear component satisfies the equation

$$\frac{d^2\tilde{u}_1}{dr^2} \;+\; \frac{2}{r}\frac{d\tilde{u}_1}{dr} \;+\; \left(k^2 \;-\; \frac{2}{r^2}\right)\tilde{u}_1 \;=\; f \tag{39}$$

which is similar to equation (25). Integrating (39) and returning to the time domain, we have for the nonlinear field

$$u_1 \;=\; -\frac{D}{\sqrt{2\pi}(\lambda\;+\;2\mu)}\frac{1}{4v^4}\int_{-\infty}^{\infty} k g_1(kr) Q_2 e^{i\omega t} d\omega$$

$$-\;\frac{E}{\sqrt{2\pi}(\lambda\;+\;2\mu)}\int_{-\infty}^{\infty}\left(\frac{k}{4v^4}g_2(kr)Q_2 \;+\; \frac{k^3}{v^2}g_3(kr)Q_1 \;+\; k^5 g_4(kr)Q_0\right)e^{i\omega t}d\omega \tag{40}$$

with dimensionless functions $g_i(x)$ that have forms

$$g_1(x) \;=\; h_1^{(1)}(x)Ei(-i2x) \;+\; h_1^{(2)}(x)lnx \tag{41}$$

$$g_2(x) \;=\; h_1^{(1)}(x)Ei(-i2x) \;+\; h_1^{(2)}(x)\ln x \;+\; \frac{2i}{x^2}e^{-ix} \tag{42}$$

$$g_3(x) \;=\; \frac{1\;+\;2ix}{4x^3}e^{-ix} \tag{43}$$

$$g_4(x) \;=\; -\frac{e^{-ix}}{2x^3}\left(\frac{1}{2} \;-\; \frac{i}{x} \;-\; \frac{1}{x^2}\right) \tag{44}$$

Integrating in(40), we obtain

$$u_1 \; = \; -\frac{D \, + \, E}{\lambda \, + \, 2\mu} I_1(r) \; + \; \frac{E}{\lambda \, + \, 2\mu} I_2(r) \tag{45}$$

where

$$I_1(r) \; = \; \frac{1}{4rv^4} \left(q_{,tt}^2(t_0) \ln r \; + \; \int_r^\infty q_{,tt}^2(t + \frac{r}{v} - \frac{2x}{v}) \left(\frac{2lnx}{r} \; - \; \frac{1}{x} \right) dx \right) \tag{46}$$

$$I_2(r) \; = \; \frac{2q_{,tt}(t_0)q(t_0) + 3q_{,t}^2(t_0)}{4v^2r^3} \; + \; \frac{q_{,tt}(t_0)q_{,t}(t_0)}{v^3r^2}$$

$$+ \; \frac{q(t_0)q_{,t}(t_0)}{vr^4} \; + \; \frac{q^2(t_0)}{2r^5} \; - \; \frac{1}{2v^3r^2} \int_{-\infty}^{t_0} q_{,tt}^2(\tau) d\tau, \quad t_0 \; = \; t - \frac{r}{v} \tag{47}$$

An arbitrary field of the form (34) may be added to solution (45) to match the boundary conditions. This specifically enables us to represent equation (45) in the form

$$I_1(r) \; = \; \frac{1}{4rv^4} \left(q_{,tt}^2(t_0) ln \frac{r}{R} \; + \; v \int_{-\infty}^{t_0} q_{,tt}^2(\tau) \left[\frac{1}{r} \ln \left(\frac{vt + r - v\tau}{2R} \right) \; - \; \frac{1}{vt + r - v\tau} \right] d\tau \right), \tag{48}$$

where the far-field part of the nonlinear field becomes equal to zero at $r \, = \, R$. Therefore, in the far-field approximation, we have

$$u_1 \; = \; -\frac{(3\lambda \, + \, 6\mu \, + \, 2A \, + \, 6B \, + \, 2C)}{4(\lambda \, + \, 2\mu)v^4} \frac{\ln(r/R)}{r} q_{,tt}^2(t - \frac{r}{v}). \tag{49}$$

The obtained forms for nonlinear field make it equal to zero at the reference radius $r \, = \, R$.

It is a quite common case when the velocity of the displacement, rather than displacement itself is being recorded. Taking the time derivative of the field (45), we have for the velocity of the total-field displacement

$$u_{,t} \; = \; \frac{q_{,tt}(t_0)}{vr} \; + \; \frac{q_{,t}(t_0)}{r^2} \; =$$

$$-\frac{D \, + \, E}{\lambda \, + \, 2\mu} \frac{1}{2rv^4} \left(q_{,tt}(t_0) q_{,ttt}(t_0) lnr \; + \; \frac{v}{r}(1 - 2lnr) q_{,tt}^2(t_0) \; + \; v \int_r^\infty q_{,tt}^2(t + \frac{r}{v} - \frac{x}{2v}) \left(\frac{2}{rx} + \frac{1}{x^2} \right) dx \right)$$

$$-\frac{E}{\lambda + 2\mu}\left(\frac{2q_{,ttt}(t_0)q_{,t}(t_0) + q_{,tt}^2(t_0)}{2v^3r^2} + \frac{q_{,ttt}(t_0)q(t_0) + 4q_{,tt}(t_0)q_{,t}(t_0)}{2v^2r^3} + \frac{q_{,tt}(t_0)q(t_0) + q_{,t}^2(t_0)}{vr^4} + \frac{q(t_0)q_{,t}(t_0)}{r^5}\right) \tag{50}$$

Obtained analytical solutions for equation (39) are verified by comparison with its finite-difference solutions. Expressions for the source function q_τ can be found in Appendix B.

6 Discussion

Obtained results for nonlinear field components show a complex dependence of near-field terms on propagation distance. If field measurements are taken close to powerful sources (such as explosions), then this complexity should be taken into account when estimating medium parameters (nonlinear constants and attenuation).

If the condition

$$\beta_0 = \frac{D + E}{\lambda + 2\mu} = \frac{3(\lambda + 2\mu) + 4m + 2l}{\lambda + 2\mu} = 0 \tag{51}$$

is satisfied, then the nonlinear field vanishes in the far field. Using data from Huges and Kelly (1953) for solid materials, we have $\beta_0 = -6.5$ for polystyrene, $\beta_0 = -7.3$ for armco iron, $\beta_0 = 4.4$ for pyrex. Observations in rock showed much higher nonlinear coefficients. For sandstone, β_0 can be as high as 7000 (Johnson et al.,1993). These evaluations, however, were obtained for small strains on the order of $10^{-5} - 10^{-4}$. Seismic waves caused by large explosions and earthquakes strains have the order of $10^{-3} - 10^{-2}$.

7 Conclusions

Solutions have been presented for spherical wave propagation in cases involving the harmonical time dependence of a primary wave and a pulse. It was shown that a nonlinear component in the far field has resonant character and $\ln r/r$ dependence as propagation distance r grows. A nonlinear field gives relative enhancement to both low-frequency and high-frequency parts of the total field. The near-field component of the primary field does not generate a far field of the nonlinear component (in the approximation considered). A numerical approach is required to reveal the details of the nonlinear field generation.

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Appendix A

The following equation

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + \left(k^2 - \frac{2}{r^2}\right)u = \frac{e^{-iz}}{r^n}\,, \qquad z = kr \tag{A1}$$

has the following solutions:

$$u = -\frac{i}{2}\left(h_1^{(1)}(z)Ei(-i2z) + h_1^{(2)}(z)\ln z + \frac{2i}{z^2}e^{-iz}\right), \qquad \text{for} \quad n = 2 \tag{A2}$$

$$u = -\frac{ik}{2}\frac{e^{-iz}}{z^2}\,, \qquad \text{for} \quad n = 3 \tag{A3}$$

$$u = -\frac{ik^2}{6}\left(2h_1^{(1)}(z)Ei(-i2z) + \frac{i}{z^2}e^{-iz}\right), \qquad \text{for} \quad n = 4 \tag{A4}$$

$$u = \frac{ik^3}{2}\left(\frac{2i}{3}h_1^{(1)}(z)Ei(-i2z) - \left(\frac{i}{2} + \frac{z}{3}\right)\frac{e^{-iz}}{z^3}\right), \qquad \text{for} \quad n = 5 \tag{A5}$$

$$u = \frac{ik^4}{10}\left(2h_1^{(1)}(z)Ei(-i2z) + \left(iz - \frac{3}{2} - \frac{i}{z}\right)\frac{e^{-iz}}{z^3}\right), \qquad \text{for} \quad n = 6 \tag{A6}$$

$$u = \frac{ik^5}{18}\left(-\frac{i8}{5}h_1^{(1)}(z)Ei(-i2z) + \left(\frac{4z}{5} + \frac{6i}{5} - \frac{4}{5z} - \frac{i}{z^2}\right)\frac{e^{-iz}}{z^3}\right), \qquad \text{for} \quad n = 7 \tag{A7}$$

with first index spherical Hankel functions

$$h_1^{(1)}(x) = -\frac{i + x}{x^2}e^{ix}\,, \quad h_1^{(2)}(x) = \frac{x - i}{x^2}e^{-ix} \tag{A8}$$

of the first and second kind, respectively. Function $Ei(-ix)$ is defined by

$$Ei(-ix) \;=\; -\int_x^\infty \frac{e^{-i\xi}}{\xi} d\xi \tag{A9}$$

Appendix B

For a step pressure source when the pressure inside of a spherical cavity has the form

$$P(t) \;=\; P_0 U(t), \tag{B1}$$

where $U(t)$ is a unit step (Heavyside) function, we have the following expressions for $q(\tau)$ and its derivatives

$$q(\tau) \;=\; U(\tau)\frac{P_0 R^3}{4\mu}\left(1 \;-\; e^{-\eta\frac{v}{R}\tau}\left[\cos(\omega_0\tau) \;+\; \frac{\eta}{\sqrt{2\eta-\eta^2}}\sin(\omega_0\tau)\right]\right) \tag{B2}$$

$$q_{,\tau}(\tau) \;=\; U(\tau)\frac{P_0 R^2 v}{2\mu}\frac{\eta}{\sqrt{2\eta-\eta^2}}e^{-\eta\frac{v}{R}\tau}\sin(\omega_0\tau) \tag{B3}$$

$$q_{,\tau\tau}(\tau) \;=\; U(\tau)\frac{P_0 R v^2\eta}{2\mu}e^{-\eta\frac{v}{R}\tau}\left[\cos(\omega_0\tau) \;-\; \frac{\eta}{\sqrt{2\eta-\eta^2}}\sin(\omega_0\tau)\right] \tag{B4}$$

$$q_{,\tau\tau\tau}(\tau) \;=\; U(\tau)\frac{P_0 v^3\eta^2}{\mu}e^{-\eta\frac{v}{R}\tau}\left[\frac{\eta-1}{\sqrt{2\eta-\eta^2}}\sin(\omega_0\tau) \;-\; \cos(\omega_0\tau)\right] \tag{B5}$$

Waveforms for Haskell's reduced displacement potential (Banghar,1983; Saikia et al., 2001) are described by the functions

$$q(\tau) \;=\; U(\tau)\left[1 \;-\; e^{-k\tau}\left(1 \;+\; k\tau \;+\; \frac{(k\tau)^2}{2} \;+\; \frac{(k\tau)^3}{6} \;-\; B_0(k\tau)^4\right)\right] \tag{B6}$$

$$q_{,\tau}(\tau) \;=\; U(\tau)ke^{-k\tau}\left((4B_0+\frac{1}{6})(k\tau)^3 \;-\; B_0(k\tau)^4\right) \tag{B7}$$

$$q_{,\tau\tau}(\tau) \;=\; U(\tau)k^2e^{-k\tau}\left((12B_0+\frac{1}{2})(k\tau)^2 \;-\; (8B_0+\frac{1}{6})(k\tau)^3 \;+\; B_0(k\tau)^4\right) \tag{B8}$$

$$q_{,\tau\tau\tau}(\tau) \;=\; U(\tau)k^3e^{-k\tau}\left((24B_0+1)k\tau \;-\; (36B_0+1)(k\tau)^2 \;+\; (12B_0+\frac{1}{6})(k\tau)^3 \;-\; B_0(k\tau)^4\right) \tag{B9}$$

where

$$kR \;=\; \frac{24B_0 \;+\; 1}{6B_0} \tag{B10}$$

and constant B_0 has a value in the $0.2-0.8$ range.